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An Analytic Approach to the Valuation of American Path Dependent Options

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Abstract

In this paper, we propose a general method for pricing and hedging non-standard American options. The proposed method applies to any kind of American-style contract for which the payoff function has a Markovian representation in the state space. Specifically, we obtain an analytic solution for the value and hedge parameters of path-dependent American options such as barrier options. The solution includes standard American options as a special case.

The analytic formula also allows us to identify and exploit two key properties of the optimal exercise boundary - homogeneity in price parameters and time-invariance - for American options. In addition, some new put-call "symmetry" relations are also derived. These properties suggest a new, efficient and integrated approach to pricing and hedging a variety of standard and non-standard American options. From an implementation perspective, this approach avoids the current practice of repetitive computation of option prices and hedge ratios.

Our implementation of the analytic formula for barrier options indicates that the proposed approach is both efficient and accurate in computing option values and option hedge parameters. In some cases, our method is faster by about *four* orders of magnitude than existing numerical methods with equal accuracy. In particular, the method overcomes the difficulty that existing numerical methods have in dealing with prices close to the barrier, the case where the barrier matters most.

1 Introduction

Non-standard or exotic options are widely used today by banks, corporations and institutional investors, in their management of risk. The main reason is that although standard put and call options are useful risk management tools, they may not be suitable for hedging certain types of risks. For instance, a corporation may wish to control its raw material costs by limiting the average price paid for a commodity over time (Asian options), or obtaining protection, contingent upon the price breaching a barrier (barrier options). In these and other situations, the use of standard options may involve over-hedging (i.e. providing protection against risks that need not be hedged), and hence higher costs. Consequently, the use of non-standard options may not only fit the risk to be hedged better, but also lower the hedging cost, in such cases.

Although the payoff functions of non-standard options are often not much more complex than that of a standard option, this is not true for the pricing and hedging of such options. In most cases, such as Asian, barrier and look-back options, analytic solutions are hard to come by even for European-style contracts, since their payoffs are path dependent. To the best of our knowledge, in these cases, closed-form solutions are available only for the situation where the underlying asset price follows a log-normal diffusion. Therefore, numerical schemes have to be used to calculate the option prices and hedge parameters for American-style options and even for some European-style options.

The attractiveness of the alternative numerical schemes, such as the binomial, finite-difference, simulation and recursive methods, and their many extensions, can be judged by the twin criteria of accuracy and computational efficiency. The issue of efficiency merits attention even for standard options, as demonstrated by many authors.[1] Common approaches such as the binomial and finite-difference methods become both computationally intensive, as well as subject to large errors, for some parameter values. However, the problems with the conventional numerical methods become much worse when applied to the valuation and hedging of non-standard options. This can be readily illustrated in the case of barrier options. For such options, the binomial method is subject to severe convergence problems, and consequently, can lead to huge errors even with a large number of time-steps. The reason

[1]See, for example, recent papers by Broadie and Detemple (1996), Carr and Faguet (1995), and Huang, Subrahmanyam and Yu (1996).

is that the payoff of a barrier option is very sensitive to the position of the barrier in the lattice - a "knockout" option behaves very much like a standard option when the underlying asset price is far away from the barrier, but has a near-zero "expected" payoff, when it is close to the barrier.[2] Thus, although the details vary, there is need for a general solution to the problem of pricing American path-dependent options, such as barrier options, Asian options and look-back options. The path dependency rules out a simple Markovian representation and the American feature prevents the application of Monte Carlo simulation.

Due to the above considerations, it is important to develop analytic formulae for the prices and hedge parameters of non-standard American-style options, which we try to achieve in this paper. The general formula we develop is intuitively appealing, since it decomposes the price of a non-standard American-style option into the price of its European-style counterpart and a premium associated with the early exercise feature. Alternatively, the price of a non-standard American-style option can also be broken down into the price function of a standard American-style option and a premium/discount, associated with the non-standard feature(s). For instance, the price of an American-style barrier option can be split up into the price of a standard American-style option and a discount associated with the "knock-out" feature. By using the put-call "symmetry" condition that we derive, and the well-known relationship between "up-and-out" and "up-and-in" options, we can extend our results to a whole series of barrier options. In the general case, we also derive some characteristics of the optimal exercise boundary: time invariance and monotonicity in time and the strike price.[3]

Under the assumption of a log-normal underlying asset price process, we are able to derive an analytic formula for the price of certain types of non-standard American-style options, such as American-style barrier options. The analytic formulae allow us to calculate both

[2] Boyle and Lau (1994) and Ritchken (1995) develop a restricted binomial/trinomial method to overcome the problem. However, it is still extremely difficult to achieve convergence when the barrier is close to the current price of the underlying asset. Gao (1995) proposes an "adaptive mesh" method, which overcomes some of the problems posed by the above models. Even with this modification, the computational time increases as the current underlying price gets closer to the barrier, although at a much slower pace. Further, as shown by Gao (1995), the computational intensity of these methods is proportional to the maturity and the square of the volatility. Consequently, the computation costs associated with pricing long maturity and high volatility contracts can be prohibitively high.

[3] These properties are important in the practical implementation of the method we propose, since the boundary does not have to be recomputed separately for each option.

option prices and hedge parameters efficiently and accurately. For example, in the case of American "up-and-out" options, our numerical results indicate that the analytic formula is faster than the Ritchken (1995) method by at least *four* orders of magnitude for equally accurate prices and hedge ratios, when the underlying asset price is close to the barrier. Moreover, the computational time required by the analytic approach hardly increases as the current underlying asset price gets closer to the barrier. In fact, this problem, which is endemic in the lattice methods, is completely eliminated in our formulation. This is because the optimal exercise boundary, the sufficient input function of our valuation formula, is *independent* of the current underlying price. The analytic formulae also allow us to identify and exploit two properties of the optimal exercise boundary - homogeneity in price parameters and time invariance - for American options. These two properties suggest a new, efficient, and integrated approach to pricing and hedging American options. The method proposed here shows promise in being applied to other types of path-dependent options, such as "capped" options, and look-back and Asian options, whose payoff functions have a somewhat more complex Markovian representation in the state space.

The paper is organized as follows. In Section 2, we provide a general formula for the price of non-standard American-style options. In Section 3, we focus on American-style barrier options. We derive an analytic formula for the option price and hedge parameters under the assumption that the underlying asset price process follows a geometric Brownian motion. We also discuss the implementation of the analytic formulae and present our numerical results. In Section 4, we examine the possibility of extending our method to other non-standard American options such as look-back and Asian options. Section 5 concludes the paper.

2 A Pricing Framework

In this section, we develop a general framework for pricing and hedging non-standard American options. We assume that markets are complete so we can work in the risk-neutral pricing framework.

The method proposed here is a combination of two approaches used in the literature for pricing *standard* American options.[4] The first approach, proposed by MacMillan (1986) and Barone-Adesi and Whaley (1987), involves the decomposition of the price of an American

[4]See Myneni (1992) for a survey of recent work about standard American options.

option into that of a European option and the early exercise premium. This approach was further developed and specific results were obtained for the case of the log-normal underlying price process by Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992). The other approach by Geske and Johnson (1984), Ho, Stapleton, and Subrahmanyam (1996) and others, values an American option as an infinite sum of cash flows, each of which comes from a compound option.

Following both these approaches, we value an American option as a sum of two sets of cash flows using the decomposition approach: the value of the terminal cash flow at expiration and the value of the intermediate cash flows between the valuation date and expiration date. The former represents the value of an otherwise identical European option, and the latter, the value of the exercise privilege associated with an American option. Under the risk-neutral pricing framework, the value of an American option is equal to the sum of the expectation of these cash flows discounted by the risk-free rate. The key to this line of analysis is the cash flow argument, which is intuitive, although the same result can be obtained using the traditional replication approach.

Before proceeding with the analysis, we first define our notation as follows:

c : the price of a standard European call option.

C : the price of a standard American call option.

c_j : the price of a non-standard European call option of type j.
e.g., "j = uo" denotes an "up-and-out" barrier option.

C_j : the price of a non-standard American call option of type j.

p : the price of a standard European put option.

P : the price of a standard American put option.

p_j : the price of a non-standard European put option of type j.

P_j : the price of a non-standard American put option of type j.

G : the price of a non-standard American option.

We also use a subscript "o" to denote standard options. For instance, C_o and c_o represents the price of a standard American option and a standard European call option respectively. A superscript "p" denotes the American premium due to the early exercise feature. A superscript "d" denotes the discount due to the non-standard feature. We also make some assumptions that are common in the option pricing literature as follows:

Assumption 1 *The capital market is complete and perfect. Trading takes place continuously and without transaction costs.*

Assumption 1 allows us to use the risk-neutral pricing framework proposed by Cox and Ross (1976), and formalized and extended by Harrison and Kreps (1979), and Harrison and Pliska (1981). In the analysis that follows, we work under the risk-neutral measure.

Assumption 2 *There are two tradeable assets in the market, a risky asset and a riskless asset. The continuously compounded interest rate r is constant. The risky asset pays a constant dividend yield of $\delta \geq 0$, and its price process $\{S_t; t \geq 0\}$ follows an Itô process driven by a standard Wiener increment.*

Specifically, $\{S_t; t \geq 0\}$ is assumed to be a solution to the following stochastic differential equation

$$dS_t = \mu_s(S_t, t)\, dt + \sigma_s(S_t, t)\, dW_t \tag{1}$$

where $\mu_s = S_t(r - \delta)$ and W is a one-dimensional standard Brownian motion.

Consider an American-style option on the risky asset with an expiration date T. The option is characterized by its payoff function, which is assumed to depend on both the price of the underlying asset and other characteristics ("non-standard" factors) to be discussed below. Let the event $\mathcal{E}_t$ denote the time-t event that the optimal exercise condition is satisfied at t, i.e., the exercise value of the option exceeds its "live" value. Similarly, the event $\mathcal{N}_t$ is used to characterize explicitly the non-standard features of a given contract (e.g. crossing a barrier at t for a barrier option).

Let h_t denote the payoff derived from the optimal exercise. In this paper, we are interested in the case where h_t allows a Markovian representation in a two-dimensional state space (S_t, X_t), where $X_t = X_t(\{S_s\}_{0 \leq s \leq t})$.[5] Specifically, $h_t = h(S_t, X_t)$, where $h: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}_+$. The variable X_t is a supplementary state variable, whose evolution is assumed to be described by the drift parameter μ_x and the diffusion parameter σ_x. Let $h_t^n = h^n(S_t)$ denote the payoff derived from the non-standard factors. Hence, the payoff function of the option is equal to the sum of h_t and h_t^n. For simplicity, in the following analysis, we focus on the case

[5] This is not a strong restriction, since most contracts that are commonly used in practice have this structure. In general, of course, a non-Markovian structure always has a Markovian representation in a higher-dimensional state-space.

where $h^n(S_t) = 0$. In other words, the non-standard factors do not enter the payoff function directly, although they do have impact on the optimal exercise decision. (For instance, the payoff of a barrier option when knocked out is the rebate.)

As demonstrated in McKean (1965) and van Moerbeke (1976), the American option problem is a free-boundary problem. Following this approach, we assume that there exists an optimal exercise boundary $\mathcal{B} \equiv \{B_t; t \in [0,T]\}$ (to be specified later), above/below which an American call/put option would be exercised early.[6] The optimal exercise boundary divides the domain into two regions, the continuation region, $\mathcal{C}$, in which early exercise is not optimal, and the stopping region, $\mathcal{S}$, in which it is optimal.[7]

Under this formalization of the free-boundary problem, the option pricing function, defined as $G(S_t, X_t, t)$, is the solution to the following free-boundary problem (under certain regularity conditions):

$$\begin{aligned}
(\mathcal{D}^{s,x} - r)\, G(S_t, X_t, t) &= 0 && \forall (S,X,t) \in \mathcal{C} && (2)\\
G(S_T, X_T, T) &= h_T && && (3)\\
G(S_t, X_t, t) &> h_t && \forall (S,X,t) \in \mathcal{C} && (4)\\
G(S_t, X_t, t) &= h_t && \forall (S,X,t) \in \mathcal{S} && (5)\\
\left(\frac{\partial}{\partial S_t}, \frac{\partial}{\partial X_t}\right) G(S_t, X_t, t) &= \left(\frac{\partial}{\partial S_t}, \frac{\partial}{\partial X_t}\right) h_t && \forall (S,X,t) \in \partial\mathcal{C} && (6)
\end{aligned}$$

where the operator $\mathcal{D}^{s,x}$ is defined as follows[8]

$$\mathcal{D}^{s,x} = \frac{\partial}{\partial t} + \mu_s \frac{\partial}{\partial S} + \mu_x \frac{\partial}{\partial X} + \frac{\sigma_s^2}{2}\frac{\partial^2}{\partial S^2} + \rho_{s,x}\sigma_s\sigma_x \frac{\partial^2}{\partial S \partial X} + \frac{\sigma_x^2}{2}\frac{\partial^2}{\partial X^2}. \tag{7}$$

As mentioned before, intuition suggests that the value of any American option comes from two types of future cash flows. The cash flow at expiration represents a European option. The cash flow between the valuation date and expiration come from the exercise of the option (the early exercise premium). As a result, the price of a (non-standard) American-style option

[6] Note that the boundary $\mathcal{B}$ for a put is different from that for a call. For technical reasons, $\mathcal{B}$ is also assumed to be smooth and continuous. One may prove continuity following Jacka (1991). We do not pursue this issue here. Also, the boundary may have a dimensionality greater than one, since the optimal exercise decision may depend, in general, on (S_t, X_t).

[7] For barrier options, $\mathcal{C}$ and $\mathcal{S}$ should belong to the non-knock-out region.

[8] The correlation coefficient $\rho_{s,x}$ is one in the single-factor model examples considered in this paper. In more general cases, the correlation coefficient would be different from one.

can be written as

$$G = e^{-rT} E[\mathrm{CF}_T I_{\{\mathcal{E}_T, \mathcal{N}_T\}}] + \int_0^T e^{-rt} E[\mathrm{CF}_t I_{\{\mathcal{E}_t, \mathcal{N}_t\}}] dt \tag{8}$$

where $\{\mathrm{CF}_t, 0 \leq t \leq T\}$ represents the cash flows at time t, $E[\cdot]$ is the expectation under the risk-neutral measure, and $I_{\{\cdot\}}$ is an indicator function. We can now prove a general result for the price of a non-standard American-style option.

Theorem 1 *a) Consider a (non-standard) American-style option, whose payoff upon exercise is $h(S_t, X_t) \geq 0, \forall t \in [0, T]$. The price of the option is given by*

$$G = E[e^{-rT} h(S_T, X_T) I_{\{\mathcal{E}_T, \mathcal{N}_T\}}] + \int_0^T e^{-rt} E[((r - \mathcal{D}^{s,x}) h(S_t, X_t)) I_{\{\mathcal{E}_t, \mathcal{N}_t\}}] dt. \tag{9}$$

b) Furthermore, for an option whose payoff upon exercise is equal to the payoff of a standard option, i.e., the intrinsic value

$$h(S_t, X_t) = \phi\ (S_t - K); \qquad K > 0,\ \forall\ t\ \in\ [0, T],$$

where $\phi = 1$ for a call option and -1 for a put option, the price of the option is given by

$$G = E[e^{-rT} h(S_T, X_t) I_{\{\mathcal{E}_T, \mathcal{N}_T\}}] + \int_0^T e^{-rt} E[\phi\ (\delta S_t - rK) I_{\{\mathcal{E}_t, \mathcal{N}_t\}}] dt. \tag{10}$$

Proof. See Appendix I. □

Note that the term $(r - \mathcal{D}^{s,x}) h(S_t, X_t)$ in (9) represents the intermediate cash flow or the payoff rate. This is illustrated more clearly in (10). In the case of option contracts for which the payoff can be written in terms of the price of the underlying asset at that time, exchanging cash K for the underlying asset at time t involves, over the time interval $[t, t+dt]$, a cash flow[9]

$$\mathrm{CF}_t\ dt = (\delta S_t - rK)\phi\ dt. \tag{11}$$

We can also break up the cash flows in (10) and rewrite the price of a non-standard American put option G as the price of a standard American option plus the cash flows associated with the non-standard factors (characterized by $(\mathcal{N}_t)_{0 \leq t \leq T}$). Namely,

$$G = G_o - E[e^{-rT} h(\cdot)(I_{\{\mathcal{E}_T\}} - I_{\{\mathcal{E}_T, \mathcal{N}_T\}})] - \int_0^T e^{-rt} E[\phi(\delta S_t - rK)(I_{\{\mathcal{E}_t\}} - I_{\{\mathcal{E}_t, \mathcal{N}_t\}})] dt \tag{12}$$

[9] Kim (1990) obtains the same result for the special case of standard American options with a log-normal price process for the underlying asset.

where

$$G_o = E[e^{-rT}h(\cdot)I_{\{\mathcal{E}_T\}}]] + \int_0^T e^{-rt}E[\phi(\delta S_t - rK)I_{\{\mathcal{E}_t\}}]dt \tag{13}$$

is the price function of a standard American option.[10]

We should emphasize that the pricing formula (10) applies to many types of American-style options, since it takes into account the possibility of exercise at each date as well as the whole history of prices of the underlying asset. The only restriction is that the intrinsic value at time t, should be a function only of the current price of the underlying asset, S_t.

One immediate result from (10) is that a (non-standard) American option on a non-dividend paying stock with a payoff $h_t = S_t - K$ (a call option) should never be exercised before expiration. To see this, letting $\phi = 1$, we have from (10)

$$C_j = E[e^{-rT}h_T I_{\{\mathcal{E}_T,\mathcal{N}_T\}}] + \int_0^T e^{-rt}E[(\delta S_t - rK)I_{\{\mathcal{E}_t,\mathcal{N}_t\}}]dt. \tag{14}$$

One can see that the early exercise premium is always negative when the dividend yield δ is zero. Early exercise involves the loss of both insurance value and time value of money on the strike price. Consequently, an American option whose payoff is $S_t - K$ should never be exercised before expiration when $\delta = 0$, unless there is some kind of compensation for the absence of the dividend.[11]

3 American Barrier Options

Barrier options are options that are either extinguished ("out") or established ("in"), when the price of the underlying asset crosses a particular level ("barrier"). Common examples are "down-and-out," "down-and-in," "up-and-out" and "up-and-in" options, both calls and puts. An additional feature of some barrier options is that a rebate is paid when the option is extinguished or an additional premium is due when the option is established. In the analysis here, we focus on options with a zero additional rebate/premium.[12] Barrier options are among the most common exotic options that are used in the foreign exchange, interest

[10] This is only for purpose of exposition. Strictly speaking, it is the price of an American option only when $\mathcal{E}_t$ does not include the non-standard feature.

[11] Merton (1973) first pointed this out in the case of the standard American options and the "down-and-out" options.

[12] The analysis carries over without much difficulty, but with additional detail, for other types of barrier options with a non-zero rebate/premium.

rate and equity options markets. They are used by hedgers to obtain insurance protection above or below particular levels of the price of the underlying asset. They are also used by speculators, who have a directional view, to obtain a somewhat less expensive directional play on an underlying asset. In some instances, barrier options are American-style.

Consider an American-style barrier put option on an asset with a strike price K, a barrier H, and maturity T. Suppose the underlying asset pays no dividend.[13] For an "up-and-out" put option, the exercise decision at time t is based only on the underlying asset price at the same time and hence, the exercise boundary is uni-dimensional. The non-standard feature here is that if the asset price "hits" a barrier, the option becomes worthless. Two cases are worth analyzing here:[14]

a) $H > K$ (out-of-the-money "up-and-out");

b) $H \leq K$ (in-the-money (at-the-money) "up-and-out").

We first consider case a).

Corollary 1 *The price of an American "up-and-out" put option with the barrier level $H > K$ is given by*

$$P_{\text{uo}}(S_0, K) = p_{\text{uo}}(S_0, K) + \int_0^T rK\ e^{-rt} \Pr(S_t \leq B_t; M_0^t < H)dt \tag{15}$$

where $\Pr(\cdot)$ is the risk-neutral probability, $M_{t_1}^{t_2} \equiv \sup_{t_1 \leq \tau \leq t_2} S_\tau$, and the argument (S_0, K) is used to emphasize that the option is valued at time 0 with the underlying asset price equal to S_0 and a strike price K.

The optimal exercise boundary is determined by the following condition

$$K - B_t = \lim_{S_t \downarrow B_t} P_{\text{uo}}(S_t, K); \qquad M_0^t < H, \quad \forall t \in [0, T] \tag{16}$$

Proof. We have the exercise event $\mathcal{E}_t = \{S_t \leq B_t\}$ and the time-t event $\mathcal{N}_t = \{M_0^t < H\}$, dividend $\delta = 0$, and $\phi = -1$. Substituting these into (10) yields (15). □

Notice that the integral on the right-hand-side of (15) represents the early exercise premium. Alternatively, we can use the relationship between "up-and-out" and "up-and-in" European options,

$$p_{\text{uo}}(S_0, K) = p(S_0, K) - p_{\text{ui}}(S_0, K), \tag{17}$$

[13]The case of a non-zero dividend yield is considered in the proof of the general formula in Appendix II.

[14]Note that "in-the-money" or "out-of-the-money" are not related to the usual definition where $S < K$ or $S > K$.

and rewrite (15) as follows:

$$P_{\text{uo}}(S_0, K) = P(S_0, K) - \left[p_{\text{ui}}(S_0, K) + \int_0^T rKe^{-rt}\Pr(S_t \leq B_t; M_0^t \geq H)dt\right], \qquad (18)$$

where

$$P(S_0, K) = p(S_0, K) + \int_0^T rKe^{-rt}\Pr(S_t \leq B_t)dt \qquad (19)$$

is the payoff function of a standard American put option[15]. The pricing formula (18) indicates that the price of an American "up-and-out" put option is equal to that of a standard American put option less a discount due to the "non-standard" "knock-out" feature of an "up-and-out" option [since the terms in the bracket of (18) are positive]. The discount, in turn, consists of two parts. One is the loss of a European "up-and-in" option. The other represents the loss of interest accumulated over the time when the underlying asset price stays above the barrier. Note that the decomposition shown in (15) is obtained without making any specific assumptions about the underlying price process.

The price of the call option is equal to its European price, since the dividend yield is zero here and hence, early exercise is never optimal.

Now, consider case b). Here, we have:

Corollary 2 *If the dividend yield on the underlying asset, $\delta = 0$, an in-the-money (at-the-money) "up-and-out" American put option will always be exercised before it expires.*

Proof. We prove this by stochastic dominance, i.e., we show that the above specified unconditional exercise dominates all other strategies.

We divide all possible price paths into two categories, with set $\mathcal{A}_i = \{\{S_t\}_0^T | M_0^T < H\}$ denoting the price paths which do not cross the barrier, and set $\mathcal{A}_o = \{\{S_t\}_0^T | M_0^T \geq H\}$ denoting the price paths which breach the barrier during the life of the options.

For set $\mathcal{A}_i$, if the option is only exercised in some states, the American exercise premium is $\int_0^T rKe^{-rt}E[I_{\{\mathcal{E}_t\}}]dt$. However, unconditional exercise has an American premium of $\int_0^T rKe^{-rt}dt$ because $I_{\{\mathcal{E}_t\}} = 1$. Hence, the unconditional strategy dominates the conditional strategy.

For every path in set $\mathcal{A}_o$, there is a first "hitting" time τ when the stock price first cross the barrier, i.e. $S_\tau = H$. At the first hitting time, an unexercised option will be knocked out. For an exercised option, the portfolio include an cash position of K with accumulated

[15]See Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992) and footnote 10.

interest rates and a short position in the stock. Because an unexercised position would have been knocked out, the short position in the stock should also be closed. The cost to cover the short position is H. At time τ this practice yields a total cash flow of $Ke^{r\tau} - H > 0$ for a unconditional exercised position and $\int_0^\tau rKe^{rt}E[I_{\{\mathcal{E}_t\}}]dt + max(0, K - H) < Ke^{r\tau} - H$ for an option which is only exercised in some of the states before τ. Again the unconditional strategy dominates the conditional strategy. $\square$

As in the case of any put option, early exercise allows the holder of the option to capture the time value of money on the early receipt of the exercise price, by giving up the insurance value of the option. As long as there is no insurance value, a put option should be exercised if it is in the money. This is exactly what happens in this case.

It is worth mentioning that Corollary 2 does not carry over to the case of out-of-the-money "knock-out" options, because when $H > K$, an exercised position does not have enough cash to cover the short position in the stock when the barrier is breached. This observation also shows that the option will be exercised unconditionally, if the rebate amount is less than $K - H$.

So far, in this section, we have examined "up-and-out" put options and "down-and-out" call options. We now briefly analyze "up-and-out" call and "down-and-out" put options. Consider the case of American "up-and-out" call options. Suppose the dividend yield δ is zero. It is easy to see that, in this case, one should exercise an American "up-and-out" call option at time t only when $S_t = H$. Namely, the optimal exercise boundary coincides with the barrier. This exercise strategy is optimal as long as rebate $R \leq H - K$. However, if $R > H - K$, then one should never exercise early. In either case, though, the option value is equal to the value of a European barrier option with an effective rebate $R' = \max(R, H - K)$.

Now suppose $\delta > 0$. It is known that the terminal point B_T^c of the optimal exercise boundary of standard American call options is given by $B_T^c = \max(rK/\delta, K)$ (as shown by Kim (1990)). First, consider case (a): $\delta \leq rK/H$ or, equivalently, $rK/\delta \geq H$. In this case, the entire boundary of a standard American call option $\mathcal{B}^c \equiv (B_t^c)_{t\in[0,T]}$ will lie above the barrier since B_t^c is a decreasing function in t. The exercise boundary will again coincide with the barrier. The valuation problem is similar to that in the case of $\delta = 0$. Next, consider case (b): $\delta > rK/H$. In this case, $\mathcal{B}^c$ will intersect H. If $R \leq H - K$, the optimal exercise boundary of American "up-and-out" call options is given by $\mathcal{B}_{\text{uo}}^c \equiv (B_{\text{uo},t}^c)_{t\in[0,T]}$, where $\mathcal{B}^{c}_{uo,t} = \min(H, \mathcal{B}^{c}_{t})$ $\forall t \in [0, T]$. However, if $R > H - K$, it is optimal to hold an "up-and-out" call option and wait to be "knocked-out" in the region where $H < \mathcal{B}^{c}_{t}$. Note that this part of H is a forced-exercise boundary, *not* an optimal exercise boundary. As a result, the properties around the intersection between H and $\mathcal{B}^{c}$ are not clear and an analytical characterization of $\mathcal{B}^{c}_{uo}$ is not obvious. The valuation problem in case (b) is non-trivial even when $R \leq H - K$ where $\mathcal{B}^{c}_{uo}$ can be characterized analytically. This problem warrants separate attention, and hence, will not be pursued in this paper.[16]

American "down-and-out" put options can be analyzed in a similar fashion. The optimal exercise boundary can be characterized analytically except for the case where $rK/\delta > H$ or equivalently $\delta < rK/H$ and $R > K - H$. For instance, the exercise boundary $\mathcal{B}^{p}_{do}$ coincides with H when $\delta \geq rK/H$. Explicit pricing formulae can be also obtained in this case. The formulae with $R = K - H$ apply to American "capped" put options. However, there are no known explicit pricing formulae when $\delta < rK/H$.

In the rest of the paper, we focus on "up-and-out" put options and "down-and-out" call options.

3.1 The Lognormal Case

In this section, we make a further assumption:

Assumption 3 *Assume that the asset price process $\{S_t; t \geq 0\}$ follows a geometric Brownian motion.*[17]

One advantage of making this assumption is that we can obtain an explicit expression for the early exercise premium or the "knock-out" discount, and as a result, a quasi-closed-form solution for the price of an American "up-and-out" put option. Consequently, we can perform comparative statics analysis and examine analytically the properties of the optimal exercise

[16]Notice that when $R = H - K$, an American "up-and-out" call option is equivalent to an American "capped" call option with a cap $L = H$. Broadie and Detemple (1995) [and also Boyle and Turnbull (1989)] analyze the optimal exercise strategy for American "capped" call options and obtain an explicit formula for option prices in the case where $\delta \leq rK/L$. However, they do not report such a formula for the case where $\delta > rK/L$.

[17]The available empirical evidence suggests that that this assumption may not always be a good one. Nonetheless, the log-normal case can serve as a benchmark since the Black-Scholes model, which is based on this assumption, is widely used and understood in practice.

boundary. This has important implications for the implementation of the pricing formula. Another advantage is that there exists a put-call "symmetry" relation in the log-normal case.

We define the notation as follows:

$$\begin{aligned}
\mu &\equiv r - \sigma^2/2 \\
\lambda &\equiv \frac{r + \sigma^2/2}{\sigma^2} \\
d_2(x, y, t) &\equiv \frac{\ln(x/y) + \mu t}{\sigma\sqrt{t}} \\
d_1(x, y, t) &\equiv d_2(x, y, t) + \sigma\sqrt{t}
\end{aligned}$$

where σ denotes the volatility of the instantaneous return in the underlying asset.

3.1.1 Option Prices

The price of an American "up-and-out" put option is given by

$$P_{\mathrm{uo}}(S_0, K) = p^o_{\mathrm{uo}}(S_0, K) + P^p_{\mathrm{uo}}(S_0, K) \tag{20}$$

where p^o_{uo} and P^p_{uo} are the price of the corresponding European option and the early exercise premium respectively. Under Assumption 3, we can obtain a closed-form expression for the two terms on the right hand side of the above equation, and hence, an analytic solution (given the optimal exercise boundary) for the price of an American "up-and-out" option.

Specifically, from (17), the price of the European "up-and-out" put option can be written as [see Rubinstein and Reiner (1991)]:

$$\begin{aligned}
p^o_{\mathrm{uo}}(S_0, K) &= p^o(S_0, K) - p^o_{\mathrm{ui}}(S_0, K) \\
&= p^o(S_0, K) - (H/S_0)^{2\lambda-2} p^o(H^2/S_0, K) \qquad (21)
\end{aligned}$$

where $p^o(x, K)$ denotes the Black-Scholes price of a standard European put option with current underlying price x and strike price K. The American premium is given by

$$P^p_{\mathrm{uo}} = \int_0^T e^{-rt} rK \left[N(-d_2(S_0, B_t, t)) - (H/S_0)^{2\lambda-2} N(-d_2(H^2/S_0, B_t, t))\right] dt \tag{22}$$

thanks to the identity [see Cox and Miller (1980)]

$$\Pr(S_t \leq B_t; M_0^t < H) = N(-d_2(S_0, B_t, t)) - (H/S_0)^{2\lambda-2} N(-d_2(H^2/S_0, B_t, t)), \tag{23}$$

where $N(\cdot)$ represents the cumulative standard normal density function. Notice that the first term on the right hand side of (22) is the exercise premium of a standard American put option.[18]

For a call option on an underlying asset that pays no dividend, equation (14) indicates that

$$C_{\text{do}}(S_0, K) = c^{\text{o}}_{\text{do}}(S_0, K) - \int_0^T rKe^{-rt}\Pr(S_t \geq B_t)dt \;\leq\; c^{\text{o}}_{\text{do}}(S_0, K).$$

As argued earlier, and stated by Merton (1973) for the "down-and-out" options, an American barrier option should never be exercised before the expiration, when the dividend yield is zero.[19]

3.1.2 Put-Call "Symmetry"

McDonald and Schroder (1990) and Chesney and Gibson (1993) show that a put-call "symmetry" condition holds for standard American options.[20] Namely,

$$C(S_t, K, \delta, r) = P(K, S_t, r, \delta); \tag{24}$$

$$B^{\text{c}}_t(K, r, \delta) = \frac{K^2}{B^{\text{p}}_t(K, \delta, r)}, \tag{25}$$

where $B^{\text{c}}_t(\cdot)$ and $B^{\text{p}}_t(\cdot)$ denote the optimal exercise boundary point at time t of a standard American call and put option, respectively. We now demonstrate that a similar relation holds for American barrier options.

Theorem 2 *For barrier options there exists a put-call "symmetry" between a "down-and-out" call option and an "up-and-out" put option, i.e., the following relationships hold*

$$C_{\text{do}}(S_t, K, H, \delta, r) = P_{\text{uo}}(K, S_t, KS_t/H, r, \delta); \tag{26}$$

$$B^{\text{c}}_{\text{do},t}(K, H, r, \delta) = \frac{K^2}{B^{\text{p}}_{\text{uo},t}(K, K^2/H, \delta, r)}, \tag{27}$$

where the superscripts c *and* p *denote call and put, respectively.*

[18] It can be shown that the second term goes to zero as $H \uparrow \infty$.

[19] When $H > K$ (an "in-the-money" "down-and-out" call), one should exercise the option at H. However, the resultant payoff $H-K$ can be treated as a "rebate," and consequently, an "in-the-money" "down-and-out" call will not be exercised early either.

[20] Of course, it also holds for European options. Carr, Ellis, and Gupta (1996) demonstrate that one can use the property of put-call parity to construct a portfolio consisting of a put and a call to statically hedge European barrier options.

Proof. See Appendix. □

The intuition behind this "symmetry" is as follows. We know that the put-call "symmetry" holds for standard options. For "knock-out" options, the additional feature is the "knock-out" provision. Hence, the difference between the value of a "knock-out" option and the value of the corresponding standard option depends only on the likelihood of the asset price breaching the barrier. The likelihood of hitting the barrier is determined by the distance between the stock price and the barrier, and the drift of stock price. Under the assumption that the stock price follows log-normal diffusion, the asset price of the "down-and-out" call drifts away from the barrier at the speed of $r - \delta$. For the put option, the drift is $\delta - r$. Since the barrier is above the stock price in this case, the stock price again drifts towards the barrier at speed of $\delta - r$, in another words, away from the barrier at the speed of $r - \delta$, the same speed as in the call option case. Given the drifts in the two cases are the same, we also require that the distances between the logarithm of the stock price and the logarithm of the barrier be the same. For the call option, the distance is $\ln S - \ln H$, and for the put option, if we denote H^{p} as the barrier for it, the distance is $\ln H^{\mathrm{p}} - \ln K$. Equating the two yields

$$H^{\mathrm{p}} = \frac{SK}{H}.$$

Similar arguments also apply to the optimal boundary condition.

Note that, in principle, log-normality is a sufficient, but not a necessary condition for put-call "symmetry" to hold. However, the "symmetry" requires a strong restriction be placed on the underlying distribution even in the zero-drift case. In fact, as shown in Carr, Ellis and Gupta (1996), the diffusion term has to have a "symmetry" around the current asset price for the argument to go through.

3.1.3 Hedge Parameters

The hedge parameters can be calculated in a straightforward fashion from (20). For instance, the delta is

$$\Delta_{\mathrm{uo}} = \frac{\partial}{\partial S_0}(p^o_{\mathrm{uo}} + P^p_{\mathrm{uo}}),$$

where (for the European part)

$$\begin{aligned} \frac{\partial p^o_{\mathrm{uo}}}{\partial S_0} &= -N(-d_1(S_0, K, T)) - (H/S_0)^{2\lambda}\Big[N(-d_1(H^2/S_0, K, T)) \\ &\quad -(2\lambda - 2)p^o(H^2/S_0, K)/(H^2/S_0)\Big], \end{aligned} \tag{28}$$

and (for the American premium part)

$$\frac{\partial P^p_{\mathrm{uo}}}{\partial S_0} = -\int_0^T e^{-rt}\frac{rK}{S_0\sigma\sqrt{t}}\Big\{n(d_2(S_0,B_t,t)) + (H/S_0)^{2\lambda-2}$$
$$\Big[n(d_2(S_0,H^2/B_t,t)) - (2\lambda-2)\sigma\sqrt{t}N(-d_2(H^2/S_0,B_t,t))\Big]\Big\}\,dt. \qquad (29)$$

In the above, $n(\cdot)$ is the standard normal density function. One can show that, similar to the option price, the delta of an American "up-and-out" option also collapses to that of a standard American option as the barrier becomes infinity. Formulae for other hedge parameters (e.g. gamma) can be obtained similarly by differentiating (20) accordingly and are not presented here for brevity.

It has been generally recognized that the hedging of barrier options is more difficult than of standard options. This is mainly due to the unstable properties of the hedge parameters of the barrier options, especially near the barrier. The analytic formulae developed here allow us to analytically examine these properties and design appropriate hedging strategies.

3.1.4 The Optimal Exercise Boundary

We now examine the properties of the optimal exercise boundary. The optimal exercise boundary $\{B_t; t \in [0,T]\}$ is determined by the following condition

$$K - B_t = P_{\mathrm{uo}}(B_t, K); \qquad B_t < H, \quad \forall t \in [0,T]. \qquad (30)$$

It follows from (21) and (22) that the boundary point B_t solves

$$K - B_t = p^o_{\mathrm{uo}}(B_t,K) + \int_t^T rKe^{-r(s-t)}$$
$$\Big[N(-d_2(B_t,B_s,s-t)) - (H/B_t)^{2\lambda-2}N(-d_2(H^2/B_t,B_s,s-t))\Big]\,ds \quad (31)$$

In the context of standard American options with the log-normal underlying process, van Moerbeke (1976) proves that the exercise boundary is continuously differentiable, and Jacka (1991) and Kim (1990) discuss its monotonicity in time. We now demonstrate that the exercise boundary (for both standard and non-standard options) has two additional properties. One is homogeneity in the strike price and the barrier level and the other is time-invariance. As shown later, these two properties, combined with the fact that the boundary is independent of the underlying asset price, have important implications for the implementation of a pricing model for American options.

Theorem 3 *For American barrier options with a strike level K and a barrier H, the optimal exercise boundary is: homogeneous of degree one in the strike price K and the barrier level H (Homogeneity); invariant with respect to the time to expiration (Time-Invariance); a non-decreasing function of time, t (Monotonicity in Time); a decreasing function of the barrier level H (Monotonicity in the Barrier Level).*

Proof. See Appendix III. □

From the proofs, one can see that the time-invariance property should hold for any American option with a stationary process for the underlying asset price. The monotonicity is valid as long as the reward for stopping equals $K - S_t$. The homogeneity follows from the homogeneity of the option pricing function and relies on the log-normality assumption on the underlying process and the assumption that the payoff function $h(\cdot)$ is homogeneous of degree one in (K, H).

We state the following corollary without proof.[21]

Corollary 3 *For standard American options with a strike level K, the optimal exercise boundary is: homogeneous of degree one in the strike price K (Homogeneity); invariant with respect to the time to expiration (Time-Invariance); a non-decreasing function of time, t (Monotonicity in Time).*

Theorem 3 and Corollary 3 show the sufficiency of the log-normality assumptions for the homogeneity of the optimal exercise boundary (and the homogeneity of the option pricing function). Whereas the sufficiency has been discussed in the literature (see below), the necessity has not, to the best of our knowledge. Indeed, the homogeneity of the pricing function is sometimes *assumed* to hold in order to simplify the problem.

We now digress to discuss the implication of the homogeneity of the optimal boundary on the underlying price process. Since the homogeneity of the optimal boundary comes from the homogeneity of the option pricing function, we examine the possible restrictions on the underlying process imposed by the latter homogeneity. Furthermore, since we are interested in the necessary conditions for homogeneity, it is enough to consider the case of European options.

[21]Monotonicity in time is a previously-known result. To our best knowledge, the results on homogeneity and time-invariance have not been stated explicitly in the literature.

As shown in Merton (1973), for a standard European option, a return distribution which is independent of the initial price level is, in general, sufficient for the option price to be homogeneous of degree one in (S, K).[22] In other work, we show that this condition on the return distribution is necessary for the homogeneity of the option price in a one-factor continuous-time setting.[23]

3.2 Implementation

In this section, we first discuss the implementation of the analytic pricing formula (20). We then report some numerical results to illustrate the efficiency and accuracy of our implementation scheme. Since the procedure has been analyzed previously for standard American put options, the discussion that follows will be brief.[24]

The implementation of the analytic formula involves two steps. The first is to compute the optimal exercise boundary $\mathcal{B}$. The second is to calculate the option prices or hedge ratios taking $\mathcal{B}$ as input. Since $\mathcal{B}$ is implicitly defined by the integral equation (31), the boundary has to be calculated numerically. One numerical scheme is to compute the boundary recursively, an idea originally suggested by Kim (1990). Starting with B_T, B_{T-1} is calculated from (31). Next, B_{T-2} is calculated, also from (31), taking B_T and B_{T-1} as inputs. This procedure is repeated iteratively until the entire exercise boundary (an approximated one, strictly speaking) is generated. Once the optimal exercise boundary is obtained, the calculation of option prices and hedge ratios is straightforward, involving only a univariate numerical integration.

As in the case of standard American options, this recursive scheme, which is somewhat tedious, can be accelerated using the Richardson extrapolation scheme to compute the optimal exercise boundary. Such a scheme was first applied to the value of American options by Geske and Johnson (1984). In our case, we directly compute only a few points on the exercise boundary, and then uses the resulting option prices/hedge ratios to extrapolate the true price/hedge ratios.

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[22] Merton (1990, p.305) provides a counter example for this sufficiency condition.

[23] This points to the log-normal price process as the only known case where the option price is homogeneous of degree one in (S, K).

[24] Interested readers may refer to Huang, Subrahmanyam, and Yu (1996) for more details.

3.2.1 Optimal Exercise Boundary

As mentioned earlier, the implementation of the analytic method requires the optimal exercise boundary as an input. In this section, as an illustration, we provide plots of the optimal exercise boundary for American barrier put options. The boundary for American barrier call options can be obtained using the put-call "symmetry" relationship derived earlier. As shown below, useful information can be extracted from such a plot of the optimal exercise boundary.

Figure 1 illustrates the plots of the optimal exercise boundary for American "up-and-out" put options on non-dividend-paying stocks for different levels of the barrier. Specifically, we choose six levels of the barrier, namely, $H = 45, 45.01, 45.10, 46, 50, 100$. The values of the other relevant parameters are $K = 45$, $T - t = 1$, $\sigma = 0.2$, and $r = 0.0488$. One can see that for a given H, the exercise boundary divides the domain into two regions. The region above the boundary is called the continuation region, $\mathcal{C}$, in which exercise is not optimal, and the region below is the stopping region, $\mathcal{S}$, where it pays to exercise early. The boundary with $H = 100$ is essentially the same as the boundary of an otherwise identical standard put American option ($H = \infty$). One can see from the figure that as H decreases, *ceteris paribus*, the optimal exercise boundary moves upward, or equivalently the size of the stopping region increases. This indicates that the American feature of an "up-and-out" put option becomes more valuable as H gets higher. One interesting result obtained from plotting the optimal exercise boundary is that in the case where the dividend yield is zero, it is always optimal to exercise early an American "up-and-out" put option, when the barrier level is equal to the strike price. This is a direct result of Corollary 2. One can see from Figure 1 that the optimal exercise boundary with $H = 45 = K$ coincides with the line, $K = 45$. This implies that the "up-and-out" option should always be exercised because the setup dictates that the underlying asset price is below the strike price.

Figure 2 illustrates the price homogeneity of the optimal exercise boundary for American out-of-the-money "up-and-out" put options on non-dividend-paying stocks. Figure 2(a) shows plots of the boundary with $(K = 45, H = 46)$, the solid curve, and the boundary with $(K = 90, H = 92)$, the dashed curve, to illustrate the homogeneity in (K, H). Figure 2(b) shows plots of the boundary with $(K = 45, H = 100)$, the solid curve, and the boundary with $(K = 90, H = 500)$, the dashed curve, to illustrate the homogeneity in K when $H \gg K$.

Note that when $H \gg K$, an "up-and-out" put option is essentially equivalent to a standard American put option. So Figure 2(b) actually illustrates the homogeneity in K of optimal exercise boundaries for standard American options. The values of other relevant parameters are time to expiration $T - t = 1$ (year), volatility $\sigma = 0.2$, and risk-free rate $r = 0.0488$. In both (a) and (b), the height of the dashed curve is twice the height of the solid curve, which verifies the price homogeneity.

Figure 3 illustrates the stationarity (time-invariance) of the optimal exercise boundary of American out-of-the-money "up-and-out" put options on non-dividend-paying stocks. Two plots of the boundary are shown in the figure and differ only in time to expiration, the dashed curve with $T - t = 0.5$ (year) and the solid curve with $T - t = 1$ (year). The values of other relevant parameters are strike $K = 45$, barrier $H = 50$, volatility $\sigma = 0.2$, and risk-free rate $r = 0.0488$. When shifted to the right for $T - t = 0.5$, the dashed curve will coincide with the solid curve, which verifies the stationarity.

We should emphasize that the analytic method developed here has a definite advantage over the lattice methods in computing the optimal exercise boundary. For instance, it would be very difficult to obtain a plot as smooth as those shown in Figure 1 using a lattice method, even with a large number of time steps. Also, due to the construction of a lattice, the optimal exercise boundary in the lattice, in general, does not extend to the valuation date (i.e., it is a broken curve). In contrast, the plots shown in Figure 1, for instance, were generated using the analytic formula with 200 points (time-steps) and the amount of the computation time required is about 0.06 seconds (CPU time) on a SPARC-20 workstation.

3.2.2 Numerical Results

It has been recognized that simple binomial method is not appropriate for pricing barrier options due to the fact that the price of such options is very sensitive to the location of the barrier in the lattice. The reason for this sensitivity comes from the fact that the option-value function is not smooth around the boundary. The existence of such "kinks" and the discrete price-space in the binomial/trinomial models effectively causes a shift of the barrier to a nearby layer of nodes, once the barrier falls in-between two layer of nodes.[25]

In this section, we illustrate the accuracy and efficiency of the analytic formula (20) in

[25] For a detailed discussion about pricing errors from lattice model, see Gao (1996).

relation to some existing numerical methods which modify the standard binomial method. The two existing numerical methods that we are aware of for pricing barrier options, are the restricted binomial/trinomial methods of Boyle and Lau (1994) and Ritchken (1995), and the adaptive mesh method of Gao (1995). Boyle and Lau (1994) and Ritchken (1995) solve the problem of non-smoothness by forcing the barrier to coincide with a layer of nodes. As discussed earlier, the problem with these approaches is that as the asset price gets close to the barrier, the number of time steps needed to value this option goes to infinity. This feature renders these models difficult to apply under these circumstances. The adaptive-mesh method developed by Gao (1995) solves the problem by using a finer mesh around the barrier while maintaining a coarse structure in other places. It still suffers from the problem that the number of time steps goes to infinity as the asset price and the barrier get close to each other, although this happens only near the boundary in the time-price space, as opposed to everywhere in the restricted binomial/trinomial models. In contrast, as shown below, this sensitivity problem can be completely eliminated by using the analytic method developed here.

For a given method, the accuracy is measured by the deviation from a benchmark, more specifically by the root of the mean squared error (RMSE) or the root of the mean squared relative error (RMSRE). The benchmark is chosen to be the results from the Ritchken method with at least ten thousand time steps.[26] The efficiency is measured by the CPU time required to compute option prices or hedge ratios for a given set of contracts. We choose two sets of contracts for comparison. Each set consists of forty-eight contracts that have different values of the underlying asset price S_t at valuation date t, the time-to-expiration $T-t$, and the volatility parameter σ. The barrier level H and the strike value K are fixed at 50 and 45, respectively. The risk-free rate r is chosen to be 0.0488. In Set I, we choose $S_t = (40, 42.5, 45, 47.5)$, $T-t = (0.25, 0.5, 0.75, 1.0)$, and $\sigma = (0.2, 0.3, 0.4)$. As a result, the set of contracts include out-of-the-money, at-the-money, and in-the-money options. Set II is similar to Set I, except that those contracts with $S_t = 47.5$ are replaced by contracts with $S_t = 49.5$. The reason for this choice is to include contracts with S_t very close to the barrier.

[26] In the Ritchken method, the number of time steps cannot be chosen arbitrarily, due to the restriction that the barrier has to coincide with a node [see Ritchken (1995) for details]. In our implementation, the number of time steps used for contract Set I (to be specified later) is between $10,027$ and $11,677$, and for contract Set II (to be specified later) is between $10,027$ and $21,385$.

This is the case where the existence of a barrier matters most.

Table 1 reports numerical results of option prices and deltas from our analytic formula (20) and Ritchken (1995) method (with a minimum 50 time steps in the trinomial tree) for the two sets of contracts. Columns 2 and 3 list the results for the contracts in Set I and columns 4 and 5 for the contracts in Set II. The results for individual contracts are not shown in the table in the interest of brevity. The results about the RMSE and RMSRE for both methods are shown in the table, respectively. The CPU-times, the amount of time required (on a SPARC-20 workstation) to compute the option prices or the delta values for all the forty-eight contracts in each set, are also presented in the table.

One can see from the table that the the results from the analytic method are accurate under either of the two measures - RMSE and RMSRE - in both sets of contracts. In particular, the CPU-time required for Set II that includes the contracts with $S_t = 49.5$ very close to the barrier ($H = 50$) is the same as that for set I. This indicates that the analytic method can deal efficiently with the case in which the underlying price is very close to the barrier. The reason is that the optimal exercise boundary, the sufficient input function of the valuation formula, is *independent* of the current underlying price. As a result, the problem of the underlying price being too close to the barrier is completely avoided in our analytic approach.

The Ritchken method and our method have no apparent advantages over each other when Set I of contracts is used for comparison. The results from the former have the RMSE $= 3.348 \times 10^{-3}$, the RMSRE $= 1.252 \times 10^{-3}$, and the CPU-time 0.9166 seconds.[27] One can see that the Ritchken method is a little more accurate, but much less efficient than ours. However, when Set II of contracts is used for comparison, the CPU-time required for computing the option prices using the Ritchken method is about 128 seconds, whereas the CPU-time required using our analytic method is only about 0.35 seconds.[28]

In summary, our numerical experiments show that the analytic pricing formula (20) is

[27] In our implementation of the Ritchken's model, the number of time steps used is between 53 and 183 for Set I, and between 55 and 4,753 for Set II.

[28] We also conduct a comparison of the two methods for another set of contracts, which includes barrier options whose spot price $S = 49.75$. Numerical results indicate that it takes 2,140 seconds for the Ritchken method to price this set of options, whereas it still takes about 0.35 seconds for the analytical method to price the same set of contracts. In this case, the analytic method is faster by about *five* orders of magnitude than the Ritchken method.

both accurate and efficient. In particular, its performance is stable in the sense that both the accuracy and the efficiency are *not* sensitive to the distance between the underlying price and the barrier.

3.3 A "Tabulation" Approach to Pricing Options

The results reported in Table 1 are obtained by using a conventional implementation scheme. This scheme involves the calculation of the exercise boundary for *each* contract, i.e., for each value of the parameter set $(S_t, K, T-t, \sigma, r, H)$. However, due to its homogeneity and time-invariance properties, the exercise boundary needs to be calculated for only a few values of the parameter set. As a result, the CPU time reported in Table 1 can be reduced significantly.

The time-invariance property implies that, among all the contracts considered [characterized by the parameter set $(S_t, K, T-t, \sigma, r, H)$], only the boundary for the longest $T-t$, *ceteris paribus*, needs to be calculated. The homogeneity property suggests that among all the contracts considered, *ceteris paribus*, for standard American options only the boundary for one value of K, needs to be calculated, and for American barrier options among the contracts with the same proportional value of (K, H), only the boundary for one set of (K, H) needs to be calculated.[29]

Based on these observations, we suggest a new approach to the valuation and hedging of American options. We call it a *tabulation* approach since it calls for "tabulating" the optimal exercise boundary for different values of the parameter set $(K, T-t, \sigma, r)$. Essentially, we recommend the construction of a library (in a computer server) whose elements are the optimal exercise boundary characterized by a parameter set, say $(K = 1, T-t, \sigma, r)$, a three dimensional table for standard American options and $(K = 1, H, T-t, \sigma, r)$, a four dimensional table for barrier options. Given a complete library, computing the option prices and hedge ratios amounts to calling an "exercise boundary" function, similar to calling, say,

[29] Another advantage of computing the exercise boundary first is that, given a contract, one can easily determine if it is optimal to exercise right away at the valuation time, say t_0. Given the exercise boundary point at t_0, B_{t_0}, one would exercise the option if $S_{t_0} \leq B_{t_0}$. In contrast, the use of alternative methods would require the computation of the option value at t_0 to make the decision.

In-the-money contracts at the initial date t_0 may be rarely seen in practice. However, they may occur in simulation studies when a large sample of contracts are generated randomly and, in general, would not be filtered out by using a lattice method by which B_{t_0} is not computable.

a trigonometric function.

This tabulation approach avoids the current practice of repetitive computation of options price and hedge ratios. On a given day, traders typically need to evaluate their options positions several times. This involves computing positions of contracts written on the same underlying asset. These contracts differ only in their strike price, barrier level and time to expiration. A lattice method (that is widely used in practice) would require the construction of a tree for *each* of these contracts.[30] In contrast, the approach proposed here would require extracting only one exercise boundary since the boundary is independent of the current underlying price.

Notice that as discussed earlier, the analytic approach appears to be the only one, available so far, for computing accurately an optimal exercise boundary function. In general, the lattice methods can only generate an incomplete boundary.

4 Other American Path-Dependent Options

In this section, we discuss the possibility of applying the pricing formula (9) to other non-standard American options - look-back and Asian options. The applicability of (9) to these two kinds of options depends on whether one can obtain an explicit expression for the intermediate cash flow. In the following, we present some preliminary results on this issue.

The difficulty in pricing the American look-back and Asian options lies in the fact that the option prices are non-Markovian in the one-dimensional space spanned by the underlying price variable S_t. However, the option prices become Markovian in a two-dimensional space by using an additional state variable. This variable can be chosen to be the running minimum/maximum of S_t for the look-back options and the running average of S_t for the Asian options.

Two-factor (variable) lattice methods have been developed along these lines for such options. However, these methods still require keeping track of complex paths and hence may not be satisfactory especially given the memory limitation in computers. In contrast, the advantage of a Markovian representation can be fully captured by using the pricing

[30] Finite difference methods for solving the fundamental partial differential equation avoid the reconstruction of the lattice. However, they require interpolation to compute option prices and hedge ratios, which may not be very accurate.

formula (9). This is because the valuation problem can be reduced to a two-dimensional integration problem in (9) with the integration being Markovian. As a result, the pricing formula (9) provides a promising way to deal with the valuation problem for the path-dependent options, provided the intermediate cash flow and conditional distribution of the underlying have explicit forms.[31]

4.1 Look-back Options

There are two kinds of look-back option. One is the "standard" look-back options (or option with a flexible strike). The other is an option on extrema (or an option with fixed strike price). Closed-form solutions have been obtained for European look-back options when the underlying asset price follows a log-normal diffusion.[32] For American look-back options with a finite time-to-expiration, no closed-from solutions have been obtained so far.[33] Since such options are both path-dependent and American-style, pricing them numerically is by no means a trivial task. Babbs (1992) and Cheuk and Vorst (1994) develop a binomial method to price "standard" look-back options. However, their methods cannot be used to price options on extrema, as noted by Cheuk and Vorst (1994). A two-factor lattice method [e.g., Hull and White (1993)] may be useful in dealing with such contracts.

The pricing formula (9) developed here potentially provides an alternative method for pricing American look-back options. For instance, consider put options on extrema. The option payoff upon exercise at time t is $K - m_0^t$, where the running minimum is defined as follows

$$m_{t_1}^{t_2} \equiv \min_{t_1 \leq t \leq t_2} S_t.$$

Choosing m_0^t as the supplementary variable X_t, we can recast the problem in the framework of Theorem 1. Given the payoff structure, the optimal exercise decision variable is m_0^t. The option will be exercised at time t if m_0^t is below the optimal boundary point $B_{m,t}$. However, the exercise decision also depends indirectly on S_t. This indicates that the optimal boundary is now a two-dimensional surface in (S_t, m_0^t). As a result, the implementation in this case

[31] The intermediate cash flow can be explicitly specified for both look-back and Asian options, given the PDEs satisfied by the option pricing functions [see, for example, Wilmott, Dewynne, and Howison (1993)].

[32] See Goldman, Sosin, and Gatto (1979) and Conze and Viswanathan (1991).

[33] Duffie and Harrison (1993) obtain a closed-form solution for perpetual American lookback options with a flexible strike.

will be more involved.

4.2 Asian Options

An Asian option can be represented in a Markovian framework by introducing the running average as the supplementary state variable.[34] For geometric Asian options, the running average also follows a log-normal distribution, if the underlying price process is log-normal. In this case, the joint distribution of the spot price and the path average is known and the pricing formula (9) can be applied directly. However, for arithmetic Asian options, there is no known explicit form of the distribution function for the arithmetic average. In this case, we can still get an approximation similar to Turnbull and Wakeman (1991) by obtaining the correct mean and variance for the terminal distribution of the arithmetic average and then assuming that the average approximately follows a log-normal diffusion. Under this assumption, we can get an approximate joint distribution of the spot price and the running average and then use the state-space approach developed here.

5 Conclusion

Non-standard or exotic options are in wide-spread use today in global financial markets. Increasingly, over-the-counter options on many assets including equities, fixed income securities, foreign exchange and commodities have non-standard characteristics, such as the "knock-out"/"knock-in" feature, and the averaging of the price of the underlying asset, among others. Often, due to the lack of liquid secondary markets for such products, in view of their custom-designed nature, an optimal exercise or American-style feature is incorporated into the design of the contract. Even for standard options, the American feature causes problems for valuation and hedging, since there is no closed-form solution for the prices and hedge parameters, in general. Therefore, most models of American option valuation and hedging are implemented using numerical procedures. This problem is further compounded for non-standard American options.

The use of numerical approaches for valuation and hedging of derivatives has three prob-

[34]Hull and White (1993) develop a binomial model based on this approach to price geometric and arithmetic Asian options. Since the number of path grows too rapidly for the arithmetic case, they keep track of only a limited number of paths and use interpolation to deal with all the other paths.

lems. First, almost all the available methods are based on a lattice or grid and the accuracy of the results obtained is limited by the fineness of the grid. Even for standard American options, the errors may be large in some cases, as emphasized by many authors.[35] However, for exotic options such as barrier options, whose values are very sensitive to even minor perturbations in the parameters, the errors may be much larger. Second, the computational time necessary to reduce these errors by choosing a finer grid size may be large for exotic options. In fast-moving markets, it is obviously essential to obtain reasonably accurate prices and hedge ratios fairly quickly. Third, even if one can come up with numerical methods that are fairly efficient and accurate, it is difficult to obtain an intuitive understanding of how the pricing and hedging works, in the absence of analytical results.

These problems make it desirable, wherever possible, to derive even quasi-analytical models for non-standard American options. Our research shows that in many cases, such analytical formulae can be derived, at least for some cases of exotic options, extending the work of Kim (1990), Jacka (1991) and Carr, Jarrow and Myneni (1992). The general formulae we develop decompose the price of non-standard American option as the sum of a comparable European-style option and the optimal exercise premium.

We are able to derive analytical formulae for the prices and hedge ratios in the case of barrier options. The formulae are implemented using the Richardson extrapolation, first proposed by Geske and Johnson (1984). Our results indicate that our method is, in many cases, faster by *four* orders of magnitude and equally accurate. We present the details of our approach for American-style barrier options. We are continuing our research on the implementation of our method for the case of American-style look-back options and Asian options.

Our approach also indicates the advantage of studying the optimal exercise boundary when dealing with American options. We identify and exploit two key properties of the optimal exercise boundary - homogeneity in price parameters and time-invariance - for American options. In addition, some new put-call "symmetry" relations are also derived. These properties, in turn, suggest a new, efficient and uniform approach to pricing and hedging a variety of standard and non-standard American options. From an implementation perspective, this approach avoids the current practice of repetitive computation of option prices and

[35]See Carr and Faguet (1995), and Huang, Subrahmanyam and Yu (1996), for example.

hedge ratios.

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Appendix

I: Proof of Theorem 1

Under suitable regularity conditions, we have that [e.g. see Karatzas and Shreve (1991, p. 328)]

$$\Lambda_t = G(S_t, X_t, t)e^{-rt} - G(S_0, X_0, 0) - \int_0^t \mathcal{D}^{s,x}[e^{-ru}G(S_u, X_u, u)]du \tag{32}$$

is a martingale under the risk-neutral measure. It follows that

$$G(S_0, X_0, 0) = E_0[G(S_T, X_T, T)e^{-rT}] - \int_0^T E_0[\mathcal{D}^{s,x}[e^{-ru}G(S_u, X_t, u)]]du. \tag{33}$$

Using (2) and (5), we have

$$\mathcal{D}^{s,x}[e^{-ru}G(S_u, X_u, u)] = e^{-ru}[(\mathcal{D}^{s,x} - r)h(S_u, X_u)]I_{\{\mathcal{E}_u, \mathcal{N}_u\}}. \tag{34}$$

Applying the same argument to the terminal payoff $G(S_T, X_T, T)$ and substituting (34) into (33) yields (9) in Theorem 1. This completes the first half of the proof.

When $h_u = h(S_u) = (S_u - K)\phi$ (recall that $\phi = \pm 1$), we have

$$[(\mathcal{D}^{s,x} - r)G(S_u, X_u, u)]I_{\{\mathcal{E}_u, \mathcal{N}_u\}} = [(\mathcal{D}^s - r)h(S_u)]I_{\{\mathcal{E}_u, \mathcal{N}_u\}} = -\phi\ (\delta S_u - rK)I_{\{\mathcal{E}_u, \mathcal{N}_u\}},$$

which yields (10). This completes the second half of the proof. □

II: Proof of the Theorem 2

We prove the "put-call symmetry" for the case of the out-of-the-money "knock-out" option only. The case of the in-the-money "knock-out" can be analyzed in a similar fashion.

We define the notation first.

$$\begin{aligned} d_1(x, y, t, r, \delta) &= \frac{\ln(x/y) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}}, \\ d_2(x, y, t, r, \delta) &= \frac{\ln(x/y) + (r - \delta - \sigma^2/2)t}{\sigma\sqrt{t}}, \\ \lambda(r, \delta) &= \frac{r - \delta}{\sigma^2} + \frac{1}{2} \end{aligned}$$

To simplify notation, we shall omit the subscript "do" or "uo" and use B_t^{c} and B_t^{p} to denote the optimal exercise boundary of a "down-and-out" call option and an "up-and-out" put option, respectively.

We know that the price of a "down-and-out" American call option ($K > H$) is given by

$$C_{\rm do}(S_0, K, H, r, \delta) = c_{\rm do}(S_0, K, H, r, \delta) + C^p_{\rm do}(S_0, K, H, r, \delta),$$

where

$$\begin{aligned} c_{\rm do}(S_0, K, H, r, \delta) &= c^o(S_0, K, r, \delta) - (H/S_0)^{2\lambda(r,\delta)-2} c^o(H^2/S_0, K, r, \delta) \\ C^p_{\rm do}(S_0, K, H, r, \delta) &= \\ \int_0^T e^{-rt}\Big\{\delta S_0 &\Big[N(d_1(S_0, B_t, t, r, \delta)) - (H/S_0)^{2\lambda(r,\delta)} N(d_1(H^2/S_0, B_t, t, r, \delta))\Big] \\ &-rK\Big[N(d_2(S_0, B_t, t, r, \delta)) - (H/S_0)^{2\lambda(r,\delta)-2} N(d_2(H^2/S_0, B_t, t, r, \delta))\Big]\Big\}dt \end{aligned}$$

Similarly, the price of an "up-and-out" American put option ($K < H$) is given by

$$P_{\rm uo}(S_0, K, H, r, \delta) = p_{\rm uo}(S_0, K, H, r, \delta) + P^p_{\rm uo}(S_0, K, H, r, \delta)$$

where

$$\begin{aligned} p_{\rm uo}(S_0, K, H, r, \delta) &= p^o(S_0, K, r, \delta) - (H/S_0)^{2\lambda(r,\delta)-2} p^o(H^2/S_0, K, r, \delta) \\ P^p_{\rm uo}(S_0, K, H, r, \delta) &= \\ \int_0^T e^{-rt}\Big\{rK &\Big[N(-d_2(S_0, B_t, t, r, \delta)) - (H/S_0)^{2\lambda(r,\delta)-2} N(-d_2(H^2/S_0, B_t, t, r, \delta))\Big] \\ &-\delta S_0\Big[N(-d_1(S_0, B_t, t, r, \delta)) - (H/S_0)^{2\lambda(r,\delta)} N(-d_1(H^2/S_0, B_t, t, r, \delta))\Big]\Big\}dt \end{aligned}$$

Under the transformation

$$S_0 \to K, \quad K \to S_0, \quad \text{and } H \to \frac{KS_0}{H}, \tag{35}$$

it is easy to show, by direct substitution, that

$$c_{\rm do}(S_0, K, H, r, \delta) = p_{\rm uo}(K, S_0, \frac{KS}{H}, \delta, r).$$

This indicates that the put-call symmetry holds for the European part.

Next we will show that the premium part is also invariant under the transformation (35). Notice that under this transformation, optimal exercise boundary $B_t^{\rm p}$ for the "up-and-out" put option should be replaced by $KS_0/B_t^{\rm c}$. It is easy to show that the premium part is indeed invariant with this substitution. As a result, to complete the proof, we only have to show that $KS_0/B_t^{\rm c}$ is the optimal exercise boundary for the "up-and-out" put with the strike price S_0 and barrier $H^{\rm p} = KS_0/H$. Namely, we need to prove the following condition:

$$B_t^{\rm p}(S_0, H^{\rm p}, \delta, r) B_t^{\rm c}(K, H, r, \delta) = KS_0. \tag{36}$$

We prove this by induction. Consider $t = T$ first. Since $K > H$ implies $H^{\mathrm{p}} = KS_0/H > S_0$, we know the optimal boundary for the put and the call at maturity is

$$B_T^{\mathrm{p}} = \min(\delta S_0/r, S_0) \text{ and } B_T^{\mathrm{c}} = \max(rK/\delta, K).$$

So the condition (36) is satisfied at the maturity.

Suppose now that the condition (36) holds at time $t+1$. Consider time t. Given that B_t^C is the optimal boundary at t for the "down-and-out" call $C_{\mathrm{do}}(S_0, K, H, r, \delta)$, our goal is to show that KS_0/B_t^C is the optimal boundary at t for the "up-and-out" put $P_{\mathrm{uo}}(K, S_0, KS_0/H, \delta, r)$. For the call option, the optimal boundary satisfies

$$\begin{aligned} B_t^{\mathrm{c}} - K &= C_{\mathrm{do}}^o(B_t^{\mathrm{c}}, K, H, r, \delta) + \int_t^T e^{-r(s-t)} \\ &\Big\{\delta B_t^{\mathrm{c}} \Big[N(d_1(B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta)) - (H/B_t^{\mathrm{c}})^{2\lambda(r,\delta)} N(d_1(H^2/B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta))\Big] \\ &-rK\Big[N(d_2(B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta)) - (H/B_t^{\mathrm{c}})^{2\lambda(r,\delta)-2} N(d_2(H^2/B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta))\Big]\Big\} ds. \end{aligned} \tag{37}$$

Applying the put-call symmetry to the European part, we know

$$C_{\mathrm{do}}^o(B_t^{\mathrm{c}}, K, H, r, \delta) = P_{\mathrm{uo}}^o(K, B_t^{\mathrm{c}}, KB_t^{\mathrm{c}}/H, \delta, r)$$

From the homogeneity condition we know further that

$$C_{\mathrm{do}}^o(B_t^{\mathrm{c}}, K, H, r, \delta) = \frac{B_t^{\mathrm{c}}}{S_0} P_{\mathrm{uo}}^o(KS_0/B_t^{\mathrm{c}}, S_0, KS_0/H, \delta, r). \tag{38}$$

Let $B_t' = KS_0/B_t^{\mathrm{c}}$. Substituting (38) into (37) and then multiplying both sides of (37) by S_0/B_t^{c}, we have

$$\begin{aligned} S_0 - B_t' &= P_{\mathrm{uo}}^o(B_t', S, KS_0/H, \delta, r) + \int_t^T e^{-r(s-t)} \\ &\Big\{\delta S_0 \Big[N(d_1(B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta)) - (H/B_t^{\mathrm{c}})^{2\lambda(r,\delta)} N(d_1(H^2/B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta))\Big] \\ &-rB_t'\Big[N(d_2(B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta)) - (H/B_t^{\mathrm{c}})^{2\lambda(r,\delta)-2} N(d_2(H^2/B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta))\Big]\Big\} ds. \end{aligned} \tag{39}$$

From the assumption about the induction, we know that $B_s^{\mathrm{c}} = KS_0/B_s^{\mathrm{p}}$ for $s > t$, then

$$\ln \frac{B_t^{\mathrm{c}}}{B_s^{\mathrm{c}}} = \ln \frac{B_t^{\mathrm{c}}}{KS_0/B_s^{\mathrm{p}}} = -\ln \frac{KS_0/B_t^{\mathrm{c}}}{B_s^{\mathrm{p}}},$$

and

$$\ln \frac{H^2/B_t^{\mathrm{c}}}{B_s^{\mathrm{c}}} = \ln \frac{H^2/B_t^{\mathrm{c}}}{KS_0/B_s^{\mathrm{p}}} = -\ln \frac{(H^{\mathrm{p}})^2/B_t'}{B_s^{\mathrm{p}}}.$$

Then

$$\begin{aligned}
d_1(B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta) &= -d_2(B_t', B_s^{\mathrm{p}}, s-t, \delta, r), \\
d_1(H^2/B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta) &= -d_2((H^{\mathrm{p}})^2/B_t', B_s^{\mathrm{p}}, s-t, \delta, r), \\
d_2(B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta) &= -d_1(B_t', B_s^{\mathrm{p}}, s-t, \delta, r), \\
d_2(H^2/B_t^{\mathrm{c}}, B_s^{\mathrm{c}}, s-t, r, \delta) &= -d_1((H^{\mathrm{p}})^2/B_t', B_s^{\mathrm{p}}, s-t, \delta, r).
\end{aligned}$$

Denote $H^{\mathrm{P}} = KS_0/H$ as the barrier for the put option. Then (39) becomes

$$\begin{aligned}
S_0 - B_t' \;=\; & P_{\mathrm{uo}}^{o}(B_t', S, H^{\mathrm{p}}, \delta, r) + \int_t^T e^{-r(s-t)} \\
& \Big\{\delta S_0 \Big[N(-d_2(B_t', B_s^{\mathrm{p}}, s-t, r, \delta)) - (H^{\mathrm{p}}/B_t')^{2\lambda(\delta,r)-2} N(-d_2((H^{\mathrm{p}})^2/B_t', B_s^{\mathrm{p}}, s-t, r, \delta))\Big] \\
& -rB_t' \Big[N(-d_1(B_t', B_s^{\mathrm{p}}, s-t, r, \delta)) - (H^{\mathrm{p}}/B_t')^{2\lambda(\delta,r)} N(-d_1((H^{\mathrm{p}})^2/B_t', B_s^{\mathrm{p}}, s-t, r, \delta))\Big]\Big\} ds.
\end{aligned}$$

This equation is identical to the optimal boundary equation for the "up-and-out" put option $P_{\mathrm{uo}}(K, S_0, HS_0/H, \delta, r)$ at time t. Hence,

$$B_t^{\mathrm{p}} = B_t' = KS_0/B_t^{\mathrm{c}}$$

is the solution of the equation. This completes the proof. □

III: Proof of Theorem 3

Homogeneity: We prove this by induction in a discrete time setting. The assertion is true for B_T given the boundary condition

$$B_T = \min[\min(rK/\delta, K), H]. \tag{40}$$

Next consider B_{T-1}. Neglecting the early exercise premium, we have from (31)

$$K - B_{T-1} = p_{\mathrm{uo}}^{o}(B_{T-1}, K, H),$$

where H has been explicitly specified as an argument. One can easily see from this equation that the assertion holds for B_{T-1}. Now suppose that it holds for $(B_u)_{u \geq t}$. Consider B_{t-1}. To simplify notation, let $\theta = (K, H)$. Again, we have from (31)

$$K - B_{t-1}(\theta) = p_{\mathrm{uo}}^{o}(B_{t-1}(\theta), \theta) + f(B_{t-1}(\theta), \{B_u(\theta); u \geq t\}, \theta),$$

where $f(\cdot)$ denotes the integral on the RHS of (31). Under the transformation

$$\theta \to \alpha\theta \quad \forall \alpha \in \mathcal{R}_{++},$$

the equation for the transformed boundary point $B_{t-1}(\alpha\theta)$ becomes

$$\begin{aligned} \alpha K - B_{t-1}(\alpha\theta) &= p^o_{\mathrm{uo}}(B_{t-1}(\alpha\theta), \alpha\theta) + f(B_{t-1}(\alpha\theta), \{B_u(\alpha\theta); u \geq t\}, \alpha\theta) \\ &= \alpha\ p^o_{\mathrm{uo}}(B_{t-1}(\alpha\theta)/\alpha, \theta) + f(B_{t-1}(\alpha\theta), \{\alpha B_u(\theta); u \geq t\}, \alpha\theta) \\ &= \alpha\ p^o_{\mathrm{uo}}(B_{t-1}(\alpha\theta)/\alpha, \theta) + \alpha f(B_{t-1}(\alpha\theta)/\alpha, \{B_u(\theta); u \geq t\}, \theta), \quad (41) \end{aligned}$$

where the homogeneity of $(B_u)_{u \geq t}$ has been used in the second equality. Dividing (41) by α on both sides, we have by definition[36]

$$B_{t-1}(\alpha\theta)/\alpha = B_{t-1}(\theta),$$

which says that $B_{t-1}(\theta)$ is homogeneous of degree one in θ.

Time-Invariance: It can be easily seen from (31) that

$$\{B_u; u \in [t_1, t_2]\} = \{B_u; u \in [t_1 - \tau, t_2 - \tau]\}, \quad \forall \ \ 0 \leq \tau \leq t_1 < t_2 \leq T$$

Monotonicity in Time: Differentiating (30) with respect to t on both sides and using the fact that $\partial P_{\mathrm{uo}}/\partial t < 0$ yields $\partial B_t/\partial t > 0$.

Monotonicity in the Barrier Level: Fix time t, it is obvious that an option price is an increasing function of the barrier level, i.e.

$$\frac{\partial P_{\mathrm{uo}}(S_t, H, K)}{\partial H} > 0.$$

Given that the optimal boundary condition satisfies

$$K - B_t(H, K) = \lim_{S_t \downarrow B_t} P_{\mathrm{uo}}(S_t, H, K)$$

one can take the partial derivative on the two sides, so that

$$\frac{\partial B_t(H,K)}{\partial H} = -\lim_{S_t \downarrow B_t} \frac{\partial P_{\mathrm{uo}}(S_t, H, K)}{\partial H} < 0. \qquad \square$$

[36]The uniqueness of the boundary has been assumed implicitly. See Jacka (1991) for a proof of this uniqueness for the case of the standard American option.

Table 1

Option Prices and Delta Values of American "Up-and-Out" Put Options on Stocks

($K = \$45$; $H = \$50$; $r = 4.88\%$)

(1)	(2)	(3)	(4)	(5)
	Contract Set I		Contract Set II	
	Price	Delta	Price	Delta
	Analytical Method			
RMSE	5.1778e-03	6.9745e-03	4.8355e-03	7.4640e-03
RMSRE	2.1324e-03	1.1150e-02	2.2308e-03	1.2874e-02
CPU Time (sec)	3.4999e-01	3.4999e-01	3.4999e-01	3.4999e-01
	Ritchken's Method			
RMSE	3.348e-03	7.290e-04	3.278e-03	6.600e-04
RMSRE	1.252e-03	1.243e-03	1.096e-03	1.022e-03
CPU Time (sec)	9.166-01	9.166-01	1.287e+02	1.287e+02

Table 1 reports the results from the analytic method and the Ritchken's method (with number of time steps greater than 50) for American "up-and-out" put options on non-dividend-paying stocks for two sets of contracts. Set I includes 48 contracts, each of which has a different value of the parameter set $(S_t, T-t, \sigma)$. The domain of this parameter set is $S_t = (40, 42.5, 45, 47.5)$, $T - t = (0.25, 0.5, 0.75, 1.0)$, and $\sigma = (0.2, 0.3, 0.4)$. Set II is similar to set I, except that those contracts with $S_t = 47.5$ are replaced by contracts with $S_t = 49.5$. Columns 2 and 3 show the numerical results of option prices and delta values for contract set I. Columns 4 and 5 show the numerical results of option prices and delta values for contract set II. The second and third rows show the root of the mean squared error (RMSE) and the root of the mean squared relative error (RMSRE) for the analytic method, two measures of deviation from the benchmark - the results from the Ritchken method (with minimum 10,000 time steps). The fifth and the sixth rows show the RMSE and the RMSRE for the Ritchken's method (with number of time steps greater than 50). The CPU time is the time required for each method on a SPARC-20 workstation to compute the option prices or delta values for all the 48 contracts in a given set of contracts.

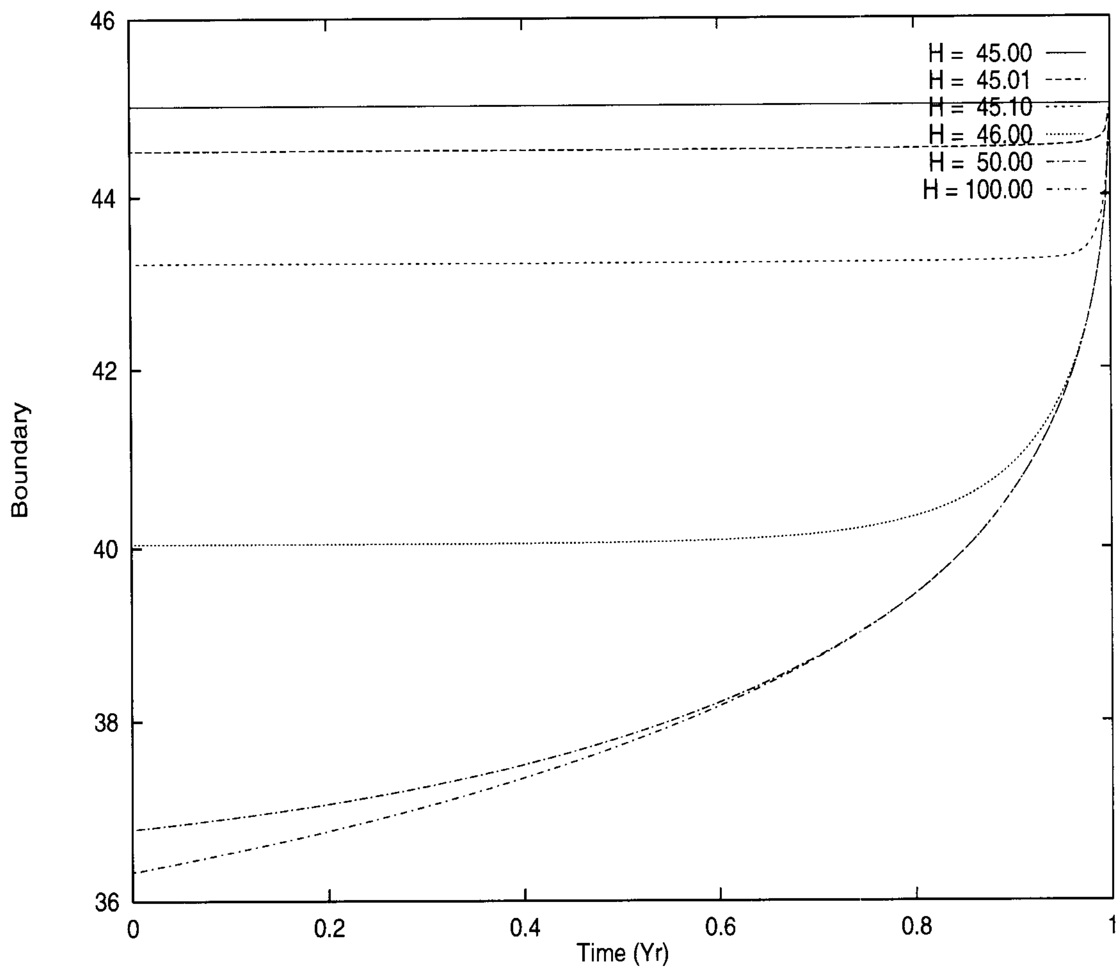
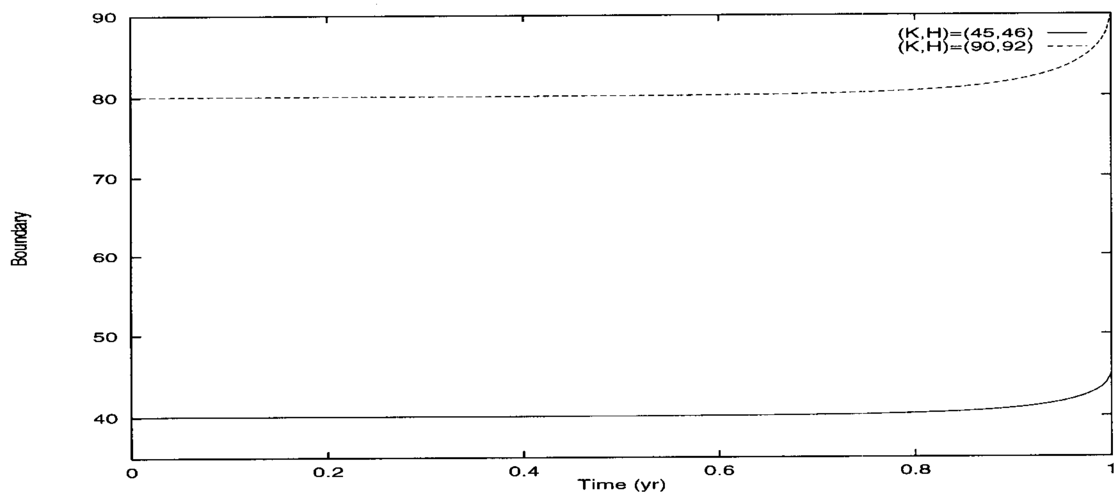
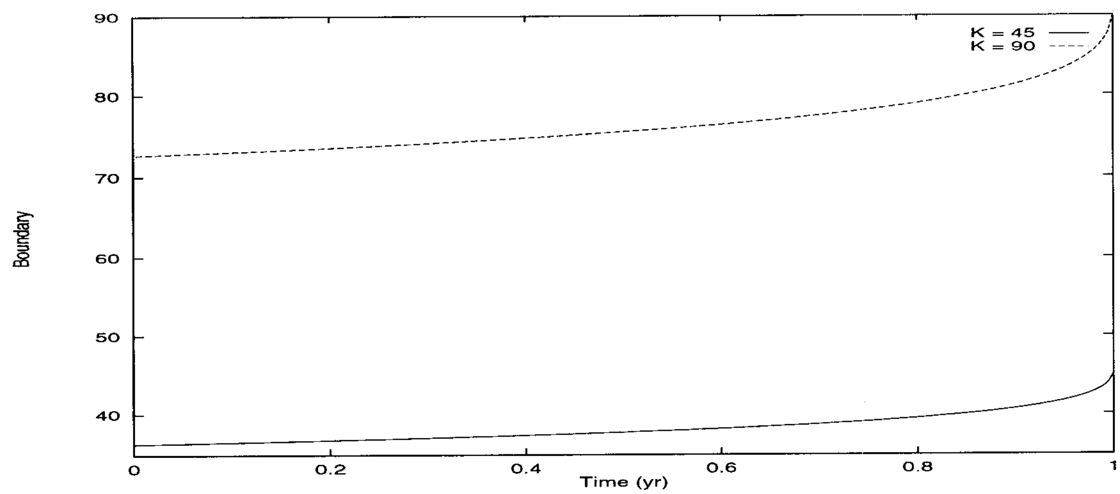


Figure 1: Optimal Exercise Boundaries of American Barrier Options with Different Barrier Levels

Figure 1 shows the plots of the optimal exercise boundary for American out-of-the-money "up-and-out" put options on non-dividend-paying stocks for different values of the barrier level H, namely $H = 45, 45.01, 45.10, 46, 50, 100$. The values of other relevant parameters are strike $K = 45$, time to expiration $T - t = 1$ (year), volatility $\sigma = 0.2$, and risk-free rate $r = 0.0488$. The boundary with $H = 45$ coincides with the strike. The boundary with $H = 100$ is essentially the same as the boundary of an otherwise identical standard American option ($H = \infty$). Each boundary shown here is generated using 200 points (time-steps).



(a)



(b)

Figure 2: Price Homogeneity of Optimal Exercise Boundaries of American "Up-and-Out" Put Options

Figure 2 illustrates the price homogeneity of the optimal exercise boundary for American out-of-the-money "up-and-out" put options on non-dividend-paying stocks. Figure 2(a) shows plots of the boundary with ($K = 45, H = 46$) and ($K = 90, H = 92$) to illustrate the homogeneity in (K, H). Figure 2(b) shows plots of the boundary with ($K = 45, H = 100$) and ($K = 90, H = 500$) to illustrate the homogeneity in K when $H \gg K$ or essentially when $H = \infty$. In both (a) and (b), the height of the dashed curve is twice the height of the solid curve (homogeneity). The values of other relevant parameters are time to expiration $T - t = 1$ (year), volatility $\sigma = 0.2$, and risk-free rate $r = 0.0488$. Each boundary shown here is generated using 200 points (time-steps).

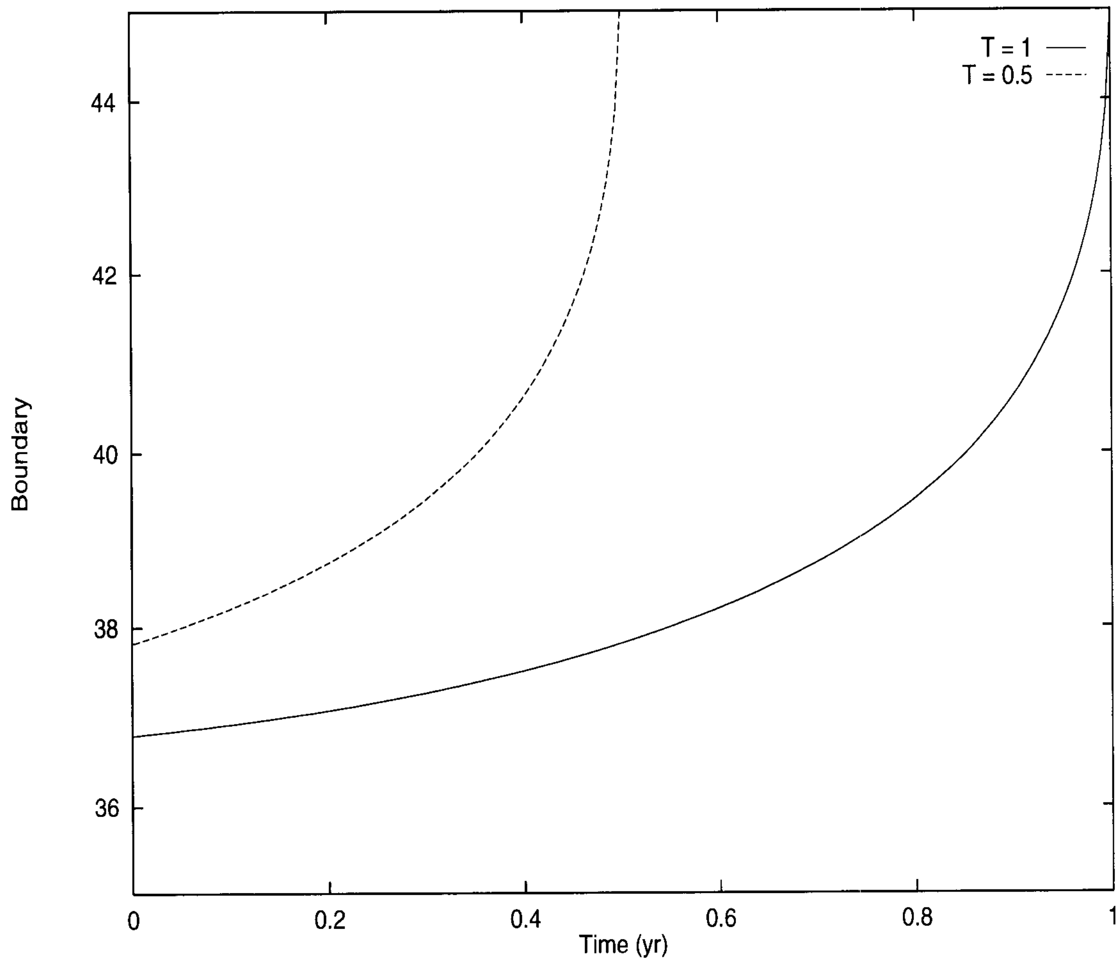


Figure 3: Stationarity of Optimal Exercise Boundaries of American "Up-and-Out" Put Options

Figure 3 illustrates the stationarity of the optimal exercise boundary of American out-of-the-money "up-and-out" put options on non-dividend-paying stocks. Two plots of the boundary are shown in the figure and differ only in time to expiration, one with $T - t = 0.5$ (year) and the other with $T - t = 1$ (year). When shifted to the right for $T - t = 0.5$, the dashed curve will coincide with the solid curve (stationarity). The values of other relevant parameters are strike $K = 45$, barrier $H = 50$, volatility $\sigma = 0.2$, and risk-free rate $r = 0.0488$. Each boundary shown here is generated using 200 points (time-steps).