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ABSTRACT:

Many valuation models in financial economics are developed using the pricing kernel approach to adjust for risk through the equivalent martingale representation. Often it is assumed, explicitly or implicitly, that the pricing kernel exhibits constant elasticity with respect to the price of the market portfolio. In a representative agent economy this would be close to assuming that the representative agent has constant proportional risk aversion. The elasticity of the pricing kernel has also implications for the pricing of options. This paper shows, first, that given the forward price of the market portfolio, all European options would have higher prices if the elasticity of the pricing kernel was declining instead of constant. Moreover, a volatility smile-effect is generated. Second, the paper shows that the standard geometric Brownian motion underlying the Black/Scholes model requires constant elasticity of the pricing kernel. Third, if the price of the market portfolio at the expiration date of an option is lognormally distributed, then declining elasticity of the pricing kernel implies a stochastic price process which is characterized by higher volatility and negative autocorrelation. Thus, declining elasticity of the pricing kernel can explain several empirical findings.

1 Introduction

Many valuation models in financial economics today are developed using the pricing kernel approach to adjust for risk, either directly, or indirectly, through the equivalent martingale representation. In such models, the value of an asset is defined as the present value of the expectation of its future payoffs weighted by the pricing kernel for end-of-period wealth. Often, it is assumed, explicitly or implicitly, that the pricing kernel exhibits constant elasticity with respect to the price of the market portfolio, i.e. a proportional change in the payoff on the market portfolio causes a constant proportional change in the pricing kernel[1]. If the constant elasticity pricing kernel is embedded in an equilibrium in a representative-agent economy, this would be equivalent to assuming that the representative agent has constant proportional risk aversion (CPRA).

The elasticity of the pricing kernel has implications for the pricing of options since it affects the pricing of claims on the market portfolio. This is the reason why most option pricing models impose an indirect restriction on the elasticity of the pricing kernel. For instance, the Black/Scholes (1973) model of option pricing, which is based on a risk-neutral valuation relationship (RNVR), is consistent with a constant elasticity pricing kernel. This is true, whether one uses the continuous trading argument of Black/Scholes (1973) and Merton (1973), or the representative-agent approach of Rubinstein (1976) and Brennan (1979). On the other hand, if the pricing kernel has non-constant (e.g., declining) elasticity, it is likely to affect the relative pricing of (extreme) claims, and hence the value of convex claims such as options.

The effect of a change in the elasticity of the pricing kernel, from constant to declining, on the pricing of claims is two-fold. First, the absolute pricing of all claims would be affected: claims that pay off in high states of the market portfolio would be valued lower and those that pay off in low states of the market portfolio would be valued higher. Second, the relative pricing of claims in the extreme states, relative to claims in the middle states, would be higher, since the "price of convexity" is increased. Thus, out-of-the-money options are priced relatively higher under the declining elasticity pricing kernel than the constant elasticity pricing kernel underlying the Black/Scholes model.

The pricing kernel is a construct than can be defined (even) in a no-arbitrage economy, and hence, it is not necessary to embed it in an equilibrium setting. However, the concept of a pricing kernel could be made more intuitive by relating it to the preferences of investors in the economy. The simplest way to do this is in a representative-agent economy. It can be shown that the elasticity of the pricing kernel is directly related to the relative risk aversion of the representative agent[2]. Previous research has shown that the assumption of a representative agent with CPRA preferences is closely related to the conditions necessary for asset prices to follow a geometric Brownian motion[3]. Thus, in this literature, CPRA turns out to

[1] See, for example, Rubinstein (1976).

[2] The analysis would be along the lines of the papers by Rubinstein (1976) and Brennan (1979).

[3] For example, Bick (1990) uses the condition that, "in an equilibrium with a representative

be a necessary condition for the price process to be consistent with the standard geometric Brownian motion assumed in the Black/Scholes model.

The question is whether the elasticity of the pricing kernel can be related to the characteristics of the price process in a no-arbitrage setting, without assuming a representative agent economy. In other words, if a standard geometric Brownian motion is assumed for the underlying asset price, a common assumption in financial models in general and the Black/Scholes model in particular, what restrictions does this impose on the behavior of the pricing kernel? Intuitively, a characteristic of the standard geometric Brownian motion is that the process generating returns should be independent of the price level. Hence, the relative price movements should be the same at low or high prices, so that the absolute price movements should be proportionately higher at high prices, relative to those at low prices. This, in turn, imposes a restriction on the pricing kernel as well as the correlation between the pricing kernel and the returns. In principle, the argument would also work in the other direction: a constant elasticity should imply that the returns are not dependent on the price level; hence, by fixing the terminal distribution to be lognormal, one can show that the return distribution follows a standard Brownian motion.

If the elasticity of the pricing kernel is declining, it exaggerates the price movements in the extreme states: in the low states, the prices would be higher, and in the high states, they would be lower, than in the case of constant elasticity. Hence, the variance of the price process would be higher when the pricing kernel has declining rather than constant elasticity. Essentially, declining elasticity of the pricing kernel results in a stochastic risk premium for the asset. Thus, in states with low cash flows, the risk premium is large, implying a relatively high expected return over the subsequent life of the asset. In states with high cash flows, the risk premium is small, and hence, the expected return is only slightly above the riskless rate of interest. This is in contrast to the constant elasticity case where the risk premium is a constant. This stochastic risk premium means that there is excess volatility induced over time, given the terminal distribution of the price of the asset. Also, if the terminal distribution is specified, the increase in the variance of the process at intermediate dates implies that the returns will exhibit negative autocorrelation: a higher price on an intermediate date will, on average, have to come down, and a lower price on an intermediate date, will, on average, have to go up.

The assumption of constant elasticity or CPRA preferences for the representative agent, and its implication of a geometric Brownian motion for returns, yields rich implications in financial economics, which are widely used. Some of these implications are for the price process of risky assets and optimal portfolio choice, and others are the pricing of options. However, some of the recorded empirical evidence is not in accord with the predictions of these models. First, the implication of CPRA prefe-

agent, the optimal policy ought to be not to trade." He defines a price process as "viable" if there exists a utility function such that an equilibrium is supported. Franke (1984) and Stapleton and Subrahmanyam (1990) use a somewhat different approach to characterize the preferences that would support a geometric random walk. They start with a process for the cash flows from the security, the fundamental exogenous variable, and derive the restrictions required for it to be transformed into a geometric random walk for prices.

rences for portfolio allocation is that individuals invest the same proportion of their total asset holdings in risky assets, independent of their wealth. This hypothesis has been rejected in cross-sectional evidence which suggests that poorer households hold a lower proportion of their wealth in risky assets compared to richer households[4]. Second, constant elasticity implies that returns follow a random walk, which is not consistent with the negative autocorrelation documented in many studies[5]. Third, the theoretical prediction of the constant elasticity model is not in line with the "excess volatility" of the stock market claimed by Shiller (1981) and Grossman and Shiller (1981).

In addition to the implications for the price process and portfolio choice, option models based on the constant elasticity assumption are not consistent with the empirical evidence. By and large, the evidence suggests that options are underpriced by the Black/Scholes model, i.e., the implied volatility of options exceeds the historical volatility of the price of the underlying asset (Canina and Figlewsky (1993)). In addition, there is evidence in support of the "smile effect", that deep-in-the-money and deep-out-of-the-money options have higher implied volatilities than options that are at-the-money (Mayhew (1995)). This is also corroborated in studies that estimate the implied pricing kernel using index options data (for example, Longstaff (1995) and Eom (1996)) which show that the distribution is leptokurtic and negatively skewed. Although many alternative explanations have been proposed for these findings, ranging from jumps in the price process to the existence of "fat tails" in the return distribution of the underlying asset, most of the explanations relate one way or another to the stochastic process followed by the price of the underlying asset.

Based on the documented empirical anomalies, it would be valid to question the basic assumption of constant elasticity of the pricing kernel and ask whether an alternative assumption would provide a better explanation for the empirical evidence. In this spirit, we provide a model based on the alternative assumption of declining elasticity. Declining elasticity of the pricing kernel, which would be consistent with decreasing proportional risk aversion of the representative agent, would be in accordance with the cross-sectional evidence on portfolio holdings, since it implies that the relative risk aversion of investors declines with wealth. Hence, individuals invest an increasing proportion of their wealth in risky assets as their wealth rises.

Since several of the anomalous empirical findings derive directly or indirectly from the price process and its implications for option pricing, we examine this issue in some detail. Given the assumption that the pricing kernel exhibits declining elasticity, we analyze the time-series properties of the endogenous price process. We find that, under this alternative assumption, the asset price follows a process with negative autocorrelation and also exhibits "excess volatility" and hence is in accord with the empirical evidence.

These implications of a pricing kernel with declining elasticity carry over to the pricing of options. When we compare the pricing of options in such an economy with that in the standard constant-elasticity setting, we need to standardize by

[4]See, for example, Cohn, Lewellen, Lease and Schlarbaum (1975), MacCrimmon and Wehrung (1986), Levy (1994) and Oehler (1995).

[5]See, for example, Poterba and Summers (1988).

holding the forward price (which "averages" the prices across future states) constant. Since the coefficient of relative risk aversion decreases with wealth, claims on the market portfolio that pay off when the cash flows on the market are low, would be underpriced by the constant elasticity model. Thus, for example, out-of-the-money put options, which pay off in the low states, would be worth more than those implied by the constant elasticity Black/Scholes model. Since the forward price is held the same, this is also true of in-the-money call options. A similar argument holds for out-of-the-money call options and in-the-money put options. We next establish, using a no-arbitrage argument that *all* options would be priced higher in the declining elasticity setting, given the same forward price. We are also able to show that the extent of overpricing relative to the constant elasticity model increases as the option moves away from the money, either more in - or out of the money, thus providing a theoretical justification for the "smile effect".

Therefore, the assumption of a pricing kernel that exhibits declining elasticity allows us to derive models that explain several empirical anomalies: characteristics of the stochastic process generating returns, such as negative serial correlation and "excess volatility", and, the pricing of options, including the "smile effect".

Section 2 presents the implication of the pricing kernel with declining instead of constant elasticity for the pricing of contingent claims. Section 3 examines the effect on the pricing kernel and the forward price process of the alternative assumption of declining elasticity, using both the continuous-time geometric Brownian motion as well as a binomial example. The next section, Section 4, relates the elasticity of the pricing kernel to investor preferences in a representative agent setting. Section 5 presents a summary of the stylized facts from the empirical evidence on the price process of the underlying asset and option prices. It explores the implications of declining elasticity for the "smile effect" in option pricing. Section 6 concludes.

2 Valuation of Contingent Claims in a No-Arbitrage Economy

2.1 Pricing of European Options

In this section, we analyze the prices of contingent claims in a perfect capital market where no arbitrage possibilities exist. We do so by examining the properties of the pricing kernel which can be used to price any claim in this economy.

Consider a date $t \in [0, T]$ where 0 is the current date and T is the terminal date. The forward price at date t, F_t, is the forward price of the (only) risky asset for delivery at time T. In the absence of arbitrage opportunities, F_t can be determined by the following relationship:

$$F_t = \mathrm{E}[F_T \, \phi_{t,T} | I_t] \tag{1}$$

where E is the expectation operator conditional on the information set at time t, I_t. F_T is the forward price at the final date, T, which is equal to the exogenously specified spot price of the asset, S_T; and $\phi_{t,T}$ is the pricing kernel, conditional on I_t, for claims paid at time T[6].

Since we consider an economy with only one risky asset, $\phi_{t,T}$ is only a function of F_T, $\phi_{t,T}(F_T)$. We assume that $\phi_{t,T}$ is strictly positive ruling out arbitrage opportunities, twice differentiable in F_T and $\partial\phi_{t,T}/\partial F_T < 0$ implying risk aversion of the economy.

A risk-free claim on a dollar to be paid at date T always has a forward price of a dollar. Therefore, by looking at forward prices instead of spot prices we can avoid specifying the risk-free rate. The no-arbitrage condition is that the pricing kernel should have an expectation of unity:

$$\mathrm{E}[\phi_{t,T}|I_t] = 1 \tag{2}$$

The forward price of the risky asset at time 0 can be written as

$$F_0 = \mathrm{E}[F_t\,\phi_{0,T}|I_0] \tag{3}$$

The same no-arbitrage pricing argument can be used to write down the forward price of a European-style contingent claim on the risky asset. If the payoff on the contingent claim at time T is $g(S_T) = g(F_T)$, the forward price of the contingent claim at time 0, C_0, is given by

$$C_0 = \mathrm{E}[g(F_T)\cdot\phi_{0,T}|I_0] \tag{4}$$

In option pricing, we generally take the price of the underlying asset as given, and consider only the *relative* pricing of the option. We take a similar approach here, F_0 is assumed to be at a given level $F_0 = F_0^*$, i. e. $F_0 = F_0^* = \mathrm{E}_0[F_T\cdot\phi_{0,T}]$. We then ask the following question. How does the forward price of the option C_0 depend on the pricing kernel, $\phi_{0,T}$, given that $F_0 = F_0^*$? Clearly, assuming $F_0 = F_0^*$ leaves much room for the shape of the pricing kernel $\phi_{0,T}$, since there are an infinite number of possible pricing kernels that satisfy the constraint. We now establish a result which characterizes the $\phi_{0,T}$ functions which satisfy equation (3).

Since option prices are dependent on the joint relationship of the pricing kernel, $\phi_{0,T}$, and the forward price on the terminal date, we can characterize option prices by the *elasticity* of the pricing kernel, $\phi_{0,T}$, with respect to the forward price on the terminal date, $F_T = S_T$. The elasticity is defined in the conventional manner as

$$\eta(F_T) = -\frac{\partial\phi_{0,T}}{\phi_{0,T}}\Big/\frac{\partial F_T}{F_T} \tag{5}$$

We define the elasticity of two different pricing kernels, both of which satisfy equation (3) as follows. The first pricing kernel $\phi_{0,T}^1$, written henceforth as ϕ_1', has constant elasticity, η_1, i.e., $\eta_1'(F_T) = 0$. The other pricing kernel $\phi_{0,T}^2$, written as ϕ_2, has declining elasticity $\eta_2(F_T)$, where $\eta_2'(F_T)$ is negative for all values of F_T. We first

[6]The pricing kernel, $\phi_{t,T}$, can also be defined by the first order condition for the optimal portfolio choice of the investor in a representative agent economy. This is discussed in more detail in section 4 below.

establish the following result about the properties of the two pricing kernels.

Lemma 1: [*Intersection of Pricing Kernels with Different Elasticities*]

Consider two pricing kernels, ϕ_1 and ϕ_2, each of which yields the same forward asset price F_0^*. Suppose for ϕ_1, the elasticity is constant, i. e. $\eta_1'(F_T) = 0$ and for ϕ_2, the elasticity is declining, i. e. $\eta_2'(F_T) < 0, \quad \forall F_T$, then the pricing kernels ϕ_1 and ϕ_2 intersect twice.

Proof: Appendix A □

The above lemma is illustrated in Figure 1. For prices below F_T^A, $\phi_2(F_T) > \phi_1(F_T)$. This implies that for contingent claims that pay off only in the region $F_T < F_T^A$, their prices will be higher under ϕ_2 than under ϕ_1. Also, for prices above F_T^B, we have $\phi_2(F_T) > \phi_1(F_T)$. Again, for contingent claims that pay off only in the region $F_T > F_T^B$, the prices will be higher under ϕ_2 than under ϕ_1.

Given that the two pricing kernels, ϕ_1 and ϕ_2, yield the same asset forward price, it follows immediately that claims that pay off in states where $F_T < F_T^A$ and $F_T > F_T^B$ would have higher prices under the declining elasticity pricing kernel, ϕ_2, than under the constant elasticity pricing kernel, ϕ_1[7]. In particular, put options with strike prices at or below F_T^A and call options with strike prices at or above F_T^B have higher prices under the declining elasticity pricing kernel. However, the put-call parity relationship relating the prices of European-style call and put options with each other, yields the following more general result which we state as Proposition 1.

Proposition 1: [*The Pricing of European-Style Options*]
Consider two pricing kernels, ϕ_1 and ϕ_2, both of which yield the same forward price of the risky asset. Suppose that for pricing kernel ϕ_1, the elasticity is constant and for pricing kernel ϕ_2, the elasticity is declining. Then, the price of any option with a positive probability of finishing out-of-the-money, is greater under pricing kernel ϕ_2 than under ϕ_1.

Proof: Appendix B □

Proposition 1 shows that given the same forward price for the underlying asset, all options, both puts and calls at *any* strike price are overpriced by the declining elasticity pricing kernel, ϕ_2, relative to the constant elasticity pricing kernel, ϕ_1[8]. The intuitive reason for this "mispricing" is that the declining elasticity pricing kernel is more convex than the one with constant elasticity. This convexity implies that convex claims, such as options, are valued higher by the declining elasticity pricing kernel, all else being the same. In other words, extreme payoffs on either side of

[7]If the pricing kernel has increasing elasticity, the results would be reversed. Although the analysis for this case would be similar, we do not consider this case since it is not in line with empirical evidence.

[8]We exclude cases where there is a zero probability of finishing out-of-the-money. For example, for a call option at a strike price of zero, the forward price is the same as the forward price of the underlying asset, and hence equal under the two pricing kernels.

the mean are priced higher by the declining elasticity pricing kernel. However, linear claims such as the forward contract on the asset are priced the same, by assumption. Although the payoffs close to the mean are priced lower by the declining pricing kernel, this is not sufficient to outweigh the overpricing of the extreme payoffs.

Proposition 1 is quite a general result for the pricing of European-style options. An important implication of this result is that option pricing models that implicitly assume a constant elasticity for the pricing kernel yield lower option prices than those that assume declining elasticity.

This "mispricing" can be illustrated with reference to the two broad classes of option pricing models that exist in the literature for the pricing of European-style contingent claims. The first class, which includes the vast majority of option pricing models, involves an assumption about the price process for the underlying asset. Usually, these models assume a continuous-time process and derive option prices using a no-arbitrage hedging argument. The most important of these is the Black and Scholes model derived assuming a geometric Brownian motion for the asset price. In the second category are models that assume a particular probability distribution for the asset price on the expiration date of the option, F_T, together with the existence of a representative agent with a given type of utility function[9].

Although models in the first category seemingly make assumptions only about the return generating process, they imply restrictions about the pricing kernel that would sustain such a process, as shown in the next section of this paper. Essentially, the assumption that the asset price follows a geometric Brownian motion holds if and only if the pricing kernel has constant elasticity. Further, if the pricing kernel has declining elasticity, the asset price process no longer follows a Brownian motion. These results suggest that in the latter setting, option prices are systematically greater than those in the Black/Scholes model.

The second class of models, that make the assumption of a representative investor, impose explicit constraints that directly translate into restrictions on the elasticity of the pricing kernel. For example, Rubinstein (1976) and Brennan (1979) assume that the representative agent has constant-proportional-risk-averse (CPRA) preferences. This implies that the pricing kernel has constant elasticity and, in turn, leads to the Black/Scholes model for option prices[10]. Again, if this assumption is replaced by one of declining elasticity, the prices are higher than in the Black/Scholes model[11].

2.2 The Volatility "Smile"

There is considerable empirical evidence that suggests that options on assets such as equities, foreign exchange, interest rates etc. at different strike prices are not priced using the same volatility. For example, the implied volatilities of options at

[9]Following up on a result in Merton (1973), Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam [(1984a) and (1984b)] and Heston (1993) developed models of this type

[10]Alternative assumptions for the utility function of the representative agent and their implications for the elasticity of the pricing kernel are discussed in section 4 below.

[11]Franke, Stapleton and Subrahmanyam (1995) used background risk to motivate the declinig elasticity of the pricing kernel.

different strike prices using the Black/Scholes model appears to produce a U-shaped curve , known as the volatility "smile". We now use the analysis to show that a smile can be supported by declining elasticity of the pricing kernel. As has been hypothesized, a smile can be generated by "fat tails" of the probability distribution of the underlying asset return. This is a rather unprecise statement because there is no mention of the pricing kernel. What actually matters is the product of the return distribution and the pricing kernel, i.e., the risk-adjusted probability distribution. To see this, consider any given return distribution and two pricing kernels as depicted in Figure 1. Since the level of the declining elasticity-kernel exceeds that of the constant elasticity-kernel for low and for high F_T-values, the risk-adjusted probability distribution under declining elasticity exhibits fat tails relative to the distribution under constant elasticity. For example, in the Black/Scholes model, the underlying asset has a lognormal distribution; a constant elasticity of the pricing kernel leads to a risk-adjusted probability distribution which still is lognormal, but the drift is changed[12]. If constant elasticity is replaced by declining elasticity, then the new risk-adjusted probability distribution is not lognormal and exhibits fat tails relative to the Black/Scholes risk-adjusted probability distribution.

Let $f_1(F_T)$ and $f_2(F_T)$ denote the risk-adjusted probability density functions of F_T under constant and declining elasticity respectively. They are both defined on the same range of F_T which is assumed to be $(0, \infty)$. Each value of F_T with positive density $f_1(F_T)$ also has positive density $f_2(F_T)$ and vice versa. $f_1(F_T)$ is assumed to be lognormal as in the Black/Scholes model. If the "true" elasticity is declining, then all option prices are determined by $f_2(F_T)$. In order to obtain the same option price using $f_1(F_T)$, the implied volatility σ_1 of the F_T-distribution has to depend on the strike price K. Hence $\sigma_1 = \sigma_1(K)$ and $f_1(F_T) = f_1(F_T, K)$; $f_1(F_T, K)$ is the implied risk-adjusted lognormal distribution for strike price K, i.e. a risk-adjusted distribution which gives the same price using the Black/Scholes model. The following proposition shows a necessary and sufficient condition for a smile.

Proposition 2: *Assume that for every strike price $K (K > 0)$ the risk-adjusted density functions $f_1(F_T, K)$ and $f_2(F_T)$ intersect at most four times. Then a smile relative to the pricing by the Black/Scholes model exists if and only if*

- - *for every $K < K^*, f_2(F_T) > f_1(F_T, K) > 0$ for $F_T \to 0$ and*

- - *for every $K > K^*, f_2(F_T) > f_1(F_T, K) > 0$ for $F_T \to \infty$.*

K^* is the strike price where the implied volatility is at a minimum.

Proof: Appendix C. □

Given the initial condition in proposition 2, the necessary and sufficient condition for a smile is that the "true" risk-adjusted probability density exceeds the implied risk-adjusted density in states in which F_T is very low or very high. Hence the "true" risk-adjusted probability density must have fat tails relative to the implied one. This condition is satisfied, subject to a qualification, if the elasticity of the "true"

[12] See Rubinstein (1976) or Brennan (1979).

pricing kernel is declining. Hence declining elasticity implies a "smile" relative to the Black/Scholes model.

The qualification needed for this statement is as follows: The elasticity of the kernel with declining elasticity, $\eta_2(F_T)$, has to exceed that of the constant elasticity kernel, η_1, by a sufficient amount in states where F_T is very low; similarly, $[\eta_1 - \eta_2(F_T)]$ has to be sufficiently high in states where F_T is very high. The reason for this qualification which is derived in Appendix D, is that the "true" and the implied probability density (before risk-adjustment) are different. Hence, the ratio of these densities changes when moving from F_T to $(F_T + dF_T)$. In order to obtain fat tails, the elasticities of the pricing kernels must differ sufficiently in the low and in the high states so as to overcompensate the change in the ratio of densities. Subject to this qualification and the assumption that the risk-adjusted density functions $f_1(F_T, K)$ and $f_2(F_T)$ intersect at most four times, declining elasticity of the pricing kernel generates a volatility smile.

3 The Relationship between the Pricing Kernel and the Price Process for the Risky Asset

3.1 Continuous Time

In this section, we consider a geometric Brownian motion for the asset price and derive its implications for the elasticity of the pricing kernel. Conversely, we analyze the case of declining elasticity of the pricing kernel and derive its implications for the price process, given the distribution of the asset price on the terminal date T.This means that (a) $\mathrm{E}(F_T|I_t) = \mathrm{E}(F_T|I_{t-\tau})\varepsilon_{t-\tau}$ with $\mathrm{E}[\varepsilon_{t-\tau}] = 0$ and $\varepsilon_{t-\tau}$ being distributed the same across dates and states and (b) the shape of the F_T-distribution is the same across dates and states with $\sigma^2(\ln F_T|I_t) = \sigma^2(\ln F_T|I_0)(1 - t/T)$.

Suppose the forward price of the risky asset for delivery at time T follows a standard geometric Brownian motion

$$\frac{dF}{F} = \mu dt + \sigma dz \tag{6}$$

where F is the forward price of the asset, μ is the instantaneous drift of the process and σ is its instantaneous standard deviation. We now state and prove a result about the implication of such a process for the properties of the pricing kernel.

<u>Proposition 3:</u>

a) A standard geometric Brownian motion of the forward price of the risky asset implies $\forall \tau, t \in [0, T]$ *that the elasticity of the pricing kernel with respect to the forward price* F_t *is constant across dates and states,* $\eta(F_t) = \mu/\sigma^2$. $\ln(F_t/F_\tau)$ *and* $\ln \phi_{\tau,t}$ *are perfectly correlated, given* F_τ.

b) Suppose that the forward price on the terminal date, F_T, *is lognormally distributed and the elasticity of the pricing kernel is constant across dates and states.*

Then, the forward price follows a standard geometric Brownian motion.

Proof:

a) Suppose the forward price process is the standard Brownian motion (6). Consider the time interval $(t, t+dt)$. Then, equations (1) and (2) can be written more generally as

$$\begin{aligned} F_t &= \mathrm{E}[F_{t+dt}\phi_{t+dt}|I_t] \\ &= \mathrm{E}[(F_t + dF_t)\phi_{t,t+dt}|I_t] \quad ; \qquad \forall t \in [0,T), \end{aligned} \tag{7}$$

$$\mathrm{E}[\phi_{t,t+dt}|I_t] = 1 \quad ; \qquad \forall t \in [0,T) \tag{8}$$

Hence, given (3), equation (7) simplifies to

$$\mathrm{E}[dF_t\phi_{t,t+dt}|I_t] = 0 \tag{9}$$

$\phi_{t,t+dt}$ is a function of $F_{t+dt} = F_t + dF_t$. Hence, by Ito's lemma, since there is no drift in $\phi_{t,t+dt}$,

$$\begin{aligned} \phi_{t,t+dt}(F_{t+dt}) &= \phi_{t,t+dt}(F_t) + \frac{\partial\phi_{t,t+dt}(F_t)}{\partial F_t}\cdot dF_t \\ &\quad + \frac{1}{2}\frac{\partial^2\phi_{t,t+dt}(F_t)}{\partial F_t^2}\cdot\sigma^2F_t^2dt \end{aligned} \tag{10}$$

Substitute $\phi_{t,t+dt}(F_{t+dt})$ from equation (10) into equation (9). Then, the product of dF_t and the last term in equation (10) can be ignored since $dtdF_t$ is of a smaller order of magnitude. Therefore, we obtain, dropping the subscript, $(\phi(F_t) \equiv \phi_{t,t+dt}(F_t)$,

$$\begin{aligned} \mathrm{E}[(\phi(F_t) + \phi'(F_t)dF_t)dF_t|I_t] &= \phi(F_t)\mu F_tdt + \phi'(F_t)\cdot\mathrm{E}[dF_t^2|I_t] \\ &= \phi(F_t)\mu F_tdt + \phi'(F_t)\cdot\sigma^2F_t^2dt \\ &= [\phi(F_t)\mu + \phi'(F_t)F_t\sigma^2]F_tdt \\ &= 0 \qquad\qquad (11) \\ \text{where } \phi'(F_t) &\equiv \frac{\partial\phi(F_t)}{\partial F_t} \end{aligned}$$

Hence, we get

$$-\frac{\phi'(F_t)}{\phi(F_t)}F_t = \eta(F_t) = \frac{\mu}{\sigma^2} \qquad \forall t \in [0,T) \tag{12}$$

Therefore, the elasticity of the pricing kernel must be equal to μ/σ^2 and, hence, be constant, since the parameters of the standard Brownian motion are specified. Let η be the constant elasticity. Then the solution of equation (12) is

$$\phi_{t,t+dt}(F_{t+dt}) = c_{t,t+dt} F^{\eta}_{t+dt}. \tag{13}$$

Since, for $\theta < \tau < t$, $\phi_{\theta,t} = \phi_{\theta,\tau}\phi_{\tau,t}$ is independent of F_τ, equation (13) implies

$$\begin{aligned} \phi_{\tau,t} &= c^*{}_{t-\tau}(F_t/F_\tau)^{\eta} \\ &\textit{or} \\ ln\phi_{\tau,t} &= lnc^*_{t-\tau} + \eta\, ln(F_t/F_\tau) \end{aligned} \tag{14}$$

with $c^*{}_{t-\tau}$ being independent of F_τ and F_t.

Therefore, $\ln\phi(\tau,t)$ and $\ln(F_t/F_\tau)$ are perfectly correlated. This proves Proposition 3a).

b) Assume that the pricing kernel has constant elasticity η across dates and states. Consider a continuous time process dF/F with instantaneous drift $a(F_t,t)$ and volatility $b(F_t,t)$. Constant elasticity implies $a(F_t,t) = \eta \cdot b(F_t,t)$. Given a lognormal distribution of F_T, the instantaneous volatility at date $(T-dt)$ is state-independent; therefore, the same is true of the instantaneous drift. Moreover, the distribution of F_T/F_{T-dt} is independent of F_{T-dt} so that F_{T-dt} is lognormally distributed. By repeating the same argument iteratively it follows that a and b are constant across all dates and states. Hence, the instantaneous distributions F_t/F_{t-dt} are state- and date-independent implying a standard geometric Brownian motion □

Proposition 3a) has two implications for the pricing kernel, and, in turn, the pricing of claims. First, the pricing kernel exhibits constant elasticity with respect to the forward price. Second, the perfect correlation between $\ln\phi_{\tau,t}$ and $\ln(F_t/F_\tau)$ implies that one can collapse the product into a single "shifted" distribution, $\ln(\hat{F}_t/F_\tau)$, where $\hat{F}_t = F_t \cdot \phi_{\tau,t}$. This is the risk-neutral valuation relationship that underlies the continuous-time Black/Scholes model or its discrete time equivalent presented by Rubinstein (1976) and Brennan (1979)[13].

Proposition 3b) shows that given a lognormal distribution of F_T, the forward price process must be a standard geometric Brownian motion if the pricing kernel has constant elasticity across states and dates. In order to obtain those results, we only assume a perfect market without arbitrage. Note that there is no need to assume an equilibrium with a representative investor as in Bick (1987), (1990).

We next derive the implication of changing the assumption of constant elasticity of the pricing kernel. We show in the case of decreasing elasticity of the pricing kernel that the variance of the forward price increases relative to the constant elasticity case and the returns exhibit negative autocorrelation[14].

Proposition 4: *Consider a continuous time economy for dates $t \in [0,T]$. Assume that the asset price at date T, F_T, is lognormally distributed. Let $F_{1,t}$ and $F_{2,t}$ be the forward prices under the constant and declining elasticity pricing kernels. Then,*

[13] Rubinstein (1976) and Brennan (1979) explicitly use a bivariate lognormal density function for $\phi_{\tau,T}$ and F_T on the terminal date and collapse it into a single risk-neutral distribution, $\hat{F}_T$.

[14] In the case of increasing elasticity of the pricing kernel, the variance declines relative to the constant elasticity case, although the returns exhibit negative autocorrelation in this case also.

a) across states, the ratio of the two prices $F_{2,t}/F_{1,t}$ increases monotonically in $F_{1,t}$ $\forall t \in (0,T)$,

b) there exists a $F^*_{1,t}$, such that

$$\begin{array}{lllll} F_{2,t} & < & [=] & [>] & F_{1,t} \quad \text{if} \\ F_{1,t} & < & [=] & [>] & F^*_{1,t} \quad \forall t \in (0,T), \end{array}$$

c)

$$\sigma^2(F_{2,t}) > \sigma^2(F_{1,t}) \qquad \forall t \in (0,T).$$

d) For dates $t = t_1, t_2, ..., t_j, ..., T$, the returns $(F_{2,t_j}/F_{2,t_j-1})$ exhibit negative autocorrelation.

Proof: Appendix E

Proposition 4 shows that a standard geometric Bownian motion is ruled out by declining elasticity. Moreover, the forward price at any intermediate date is more volatile under the declining elasticity than under the constant elasticity pricing kernel. This follows immediately from the observation that the ratio $F_{2,t}/F_{1,t}$ is monotonically increasing in $F_{1,t}$.

3.2 Discrete Time

The previous results have been derived in a continuous time setting. Similar results can be derived in a discrete time setting. In this section, we first show for a stationary binomial tree (analogous to proposition 3a)) that $\ln(F_t/F_\tau)$ and $\ln \phi_{\tau,t}$ are perfectly correlated; this implies constant elasticity of $\phi(\tau,t)$ in the limit. Second, we illustrate proposition 4 in a binomial example with three dates.

3.2.1 Implications of a Stationary Binomial Tree for the Pricing Kernel

Consider an n-stage multiplicative binomial process similar to that in Cox, Ross, Rubinstein (1979). Specifically, let u and d be the proportionate up and down movements of the binomial process over each sub-interval, then

$$\frac{F_{t+1}}{F_t} = \left\{ \begin{array}{ll} u \; , & q \\ d \; , & 1-q \end{array} \right\}, \forall t \tag{15}$$

where q is the probability of an up movement in the price over any sub-interval. When n is large, the process in (15) converges to the Brownian motion process.

Since we assume a) a single asset whose forward price moves from t to $t+1$ as a two-state branching process and b) the existence of a risk-free asset (whose forward price is a constant), we have a complete market economy. It follows that there exists a unique "risk neutral" probability measure under which the forward price of the asset is a martingale. Also the probability of an up movement under this measure over any sub-period is a constant:

$$p = \frac{1-d}{u-d} \quad ,0 \leq p \leq 1 \tag{16}$$

The forward price of the risky asset at any point of time t is hence given by the equation:

$$F_t = pF^u_{t+1} + (1-p)F^d_{t+1} \tag{17}$$

where $F^u_{t+1} = F_t u$ and $F^d_{t+1} = F_t d$ are the values of the asset in states u and d after an up movement or down movement from F_t, respectively. We now show in our setting the well-known result that the existence of the martingale measure implies the existence of a pricing kernel. We define the pricing kernel process as follows. Given the true probabilities q, $(1-q)$, (17) can be rewritten:

$$F_t = qF^u_{t+1}\left(\frac{p}{q}\right) + (1-q)F^d_{t+1}\left(\frac{1-p}{1-q}\right). \tag{18}$$

We define

$$\phi^u_{t+1} = \frac{p}{q}, \qquad \phi^d_{t+1} = \frac{1-p}{1-q} \tag{19}$$

and call ϕ_t, $t = 1, 2, \cdots, n$ the pricing kernel process. ϕ_t is a stationary, state independent process.

Perfect Correlation: In the following analysis, we use the *unconditional* pricing kernel. This may be defined as the product

$$\phi_{0,t} = \phi_1\phi_2\cdots\phi_t \tag{20}$$

We show that $\ln(F_t/F_0)$ and $\ln\phi_{0,t}$ are perfectly correlated.

We define $\phi^u = p/q$, $\phi^d = (1-p)/(1-q)$. Then if j is the number of up movements of the asset price over the period $0-t$,

$$\phi^j_{0,t} = (\phi^u)^j(\phi^d)^{n-j} \tag{21}$$

and

$$F^j_t = F_0 u^j d^{n-j} \tag{22}$$

Hence

$$\ln\phi^j_{0,t} = j\ln\phi^u + (n-j)\ln\phi^d \tag{23}$$

and

$$\ln F^j_t = \ln F_0 + j\ln u + (n-j)\ln d \tag{24}$$

Thus $\ln\phi_{0,t}$ and $\ln F_t$ are linear in j.Therefore we can write

$$\ln\phi_{0,t} = \alpha + \beta\ln(F_t/F_0) \tag{25}$$

for appropriate α and β. (25) establishes the perfect correlation of $\ln(F_t/F_0)$ and $\ln\phi_{0,t}$.

Constant Elasticity: Equation (25) is the key to understanding the restrictions imposed on the pricing kernel by the assumption of the lognormal process for the asset price. It implies that the pricing kernel has the same stochastic properties as the asset price itself, i.e. constant elasticity of the pricing kernel. In particular, in the limit as $n \to \infty$, the unconditional pricing kernel is lognormally distributed as in Rubinstein (1976) and Brennan (1979).

Although for a finite t there exists only a finite number of F_t-values, we can think of a large t so that, approximately, F_t may be considered a variable which is continuous on the range $(0,\infty)$. Then differentiating equation (25) with respect to $\ln F_t$ yields the elasticity of the pricing kernel,

$$\frac{\partial \ln \phi_{0,t}}{\partial \ln F_t} = -\eta_{0,t}(F_t) = \beta \tag{26}$$

Hence, a stationary multiplicative binomial process of F implies a constant elasticity of the pricing kernel.

3.2.2 Asset Prices in a Declining Elasticity Economy

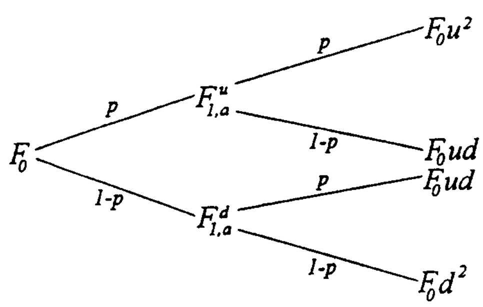
In this section, we present a simple example to illustrate the implications of a declinig elasticity of the pricing kernel in a three date-economy. We show, that in this setting, the forward price process is not a random walk, since the risk premium is stochastic, and the martingale probabilities are state dependent. Also, the asset price process exhibits "excess" volatility due to the stochastic nature of the risk premium.

Consider three points in time $(0,1,2)$. We take the time 2-price of the asset, F_2, as exogenous. We assume that the price at time 2 has a two stage binomial distribution. The states and probabilities are shown below. Note that F_0 is the forward price of the asset at time 0.

Time 2 distribution of the price F_2

state	F_2	True probability
0	F_0u^2	q^2
1	F_0ud	$2q(1-q)$
2	F_0d^2	$(1-q)^2$

Suppose, initially, that the forward price follows a stationary geometric random walk. Then we know from section 3.2.1 that there exists a martingale probability parameter p such that



where $\quad F^u_{1,a} = [pu^2 + (1-p)ud]F_0$
$F^d_{1,a} = [pud + (1-p)d^2]F_0$
$F_0 = pF^u_{1,a} + (1-p)F^d_{1,a}$

i. e. there exists a measure with constant probability parameter p under which the forward price follows a martingale.

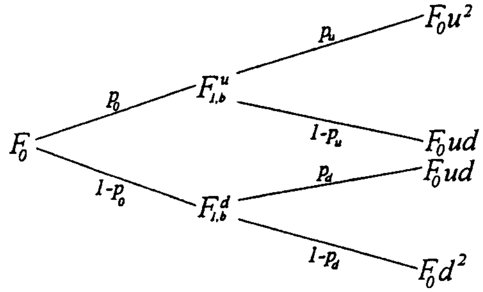
We will denote the corresponding pricing kernel process as $\phi_{1,a}(\cdot)$, $\phi_{2,a}(\cdot)$. Hence, the following equations are satisfied.

$$\begin{aligned} F_0 &= [qF^u_{1,a}\phi_{1,a}(u) + (1-q)F^d_{1,a}\phi_{1,a}(d)] \\ F^u_{1,a} &= [qF_0u^2\phi_{2,a}(uu) + (1-q)F_0ud\phi_{2,a}(ud)] \\ F^d_{1,a} &= [qF_0ud\phi_{2,a}(du) + (1-q)F_0d^2\phi_{2,a}(dd)] \end{aligned}$$

Given the results of section 3.2.1 we know in this case that

$$\begin{aligned} \phi_{1,a}(u) = \frac{p}{q} \quad &, \quad \phi_{1,a}(d) = \frac{1-p}{1-q} \\ \phi_{2,a}(uu) = \frac{p}{q} \quad &, \quad \phi_{2,a}(ud) = \frac{1-p}{1-q} \\ \phi_{2,a}(du) = \frac{p}{q} \quad &, \quad \phi_{2,a}(dd) = \frac{1-p}{1-q} \ . \end{aligned}$$

Now, consider an economy where the parameters of the pricing kernel $\phi_{t,b}()$ are such that the forward price F_0 is the same as above, but elasticity is declining. We denote the martingale probabilities in this case: p_0 is the probability of an up-move in the forward from F_0 to $F^u_{1,b}$. Then p_u and p_d are the probabilities of up-moves over the second period from $F^u_{1,b}$ and $F^d_{1,b}$ respectively.



We now establish

Corollary 2.1 *Suppose that the forward price of an asset is F_0 and at $t = 2$ it is binomially-distributed with three states F_0u^2, F_0ud and F_0d^2. Consider two pricing kernel processes $\phi_{t,a}$, $\phi_{t,b}$, each yielding the forward price F_0, where process 'a' is of the constant elasticity type and 'b' is of the declining elasticity type. Then, the martingale probabilities of an up move over the second period in the case of the declining elasticity process are $p_u > p$, $p_d < p$, where p is the* constant *martingale probability in the case of the constant elasticity process.*

Proof The two unconditional pricing kernels $\phi_{0,t,a}(\cdot) \equiv \phi_a(\cdot) = \phi_{1,a}(\cdot)\phi_{2,a}(\cdot)$ and $\phi_{0,t,b}(\cdot) \equiv \phi_b(\cdot) = \phi_{1,b}(\cdot)\phi_{2,b}(\cdot)$ must intersect twice (Lemma 1). It follows immediately that the kernels have the property:

$$\begin{aligned} \phi_a(uu) &< \phi_b(uu) \\ \phi_a(du) = \phi_a(ud) &> \phi_b(ud) = \phi_b(du) \\ \phi_a(dd) &< \phi_b(dd) \end{aligned}$$

Hence,

$$\frac{\phi_a(ud)}{\phi_a(uu)} > \frac{\phi_b(ud)}{\phi_b(uu)} \tag{27}$$

By definition, p_u is given by

$$p_u = \frac{q\phi_b(uu)}{q\phi_b(uu) + (1-q)\phi_b(ud)}$$

or

$$p_u = q\left[q + (1-q)\frac{\phi_b(ud)}{\phi_b(uu)}\right]^{-1}$$

However

$$p = q\left[q + (1-q)\frac{\phi_a(ud)}{\phi_a(uu)}\right]^{-1}$$

It follows immediately using (27) that $p_u > p$. A similar argument can be employed to establish that $p_d < p$ □

The corollary shows that the martingale probabilities in period 2 are state-dependent if the pricing kernel has declining elasticity. Hence a stationary geometric random walk of the asset forward price is ruled out. Moreover, comparing the date 1-forward prices under declining elasticity with those under constant elasticity, it follows that

$$\begin{aligned} F^u_{1,b} &= F_0u^2p_u + F_0ud(1-p_u) \\ &> F_0u^2pp + F_0ud(1-p) \\ &= F^u_{1,a} \end{aligned}$$

since $p_u > p$. Similarly, it follows that $F^d_{1,b} < F^d_{1,a}$. Hence, the volatility of the forward price $F_{1,b}$ is higher under declining elasticity.

In the C.(=constant elasticity) economy the forward price process is a stationary random walk so that the autocorrelation of the return process is zero. $F^u_{1,b} > F^u_{1,a}$ implies that the mean return in the second period, given state u, is smaller in the D (=declinig elasticity) economy than in the C economy. Conversely, $F^d_{1,b} < F^d_{1,a}$ implies that the mean return in the second period, given state d, is higher in the D economy than in the C economy. Hence we have negative autocorrelation of returns in the D economy.

These considerations show that the results in continuous- and discrete-time-models are the same, once the properties of the pricing kernel $\phi_{0,T}$ and the probability distribution of F_T are specified.

4 The Pricing Kernel and Investor Preferences

In the previous sections, it has been shown that all option prices are higher if the pricing kernel has declining instead of constant elasticity. This result is equally true in a discrete time- and a continuous time-world. In this section the relationship between the pricing kernel and investor preferences is discussed. The analysis so far has only assumed that the capital market is arbitrage-free, i.e. the pricing kernel is strictly positive everywhere.

Once investor preferences and portfolio choices come into play, it is necessary to show the existence of an equilibrium. The simplest models are those in which a representative agent with a time-additive von Neumann-Morgenstern utility function exists. Then, from the first order condition for optimal consum plan choices, the pricing kernel $\phi_{\tau t}(F_\tau, F_t)$ is determined by the marginal rate of substitution between states F_τ and F_t, deflated by the risk-free interest rate between those states. Let $u_t(C_t)$ be the utility function of the representative agent relating his date t-utility to his date t-consumption C_t. Then, his marginal rate of substitution between states F_τ and F_t is $u'_t(F_t)/u'_\tau(F_\tau)$. Let $exp[r_{t-\tau}(t-\tau) \mid I_\tau]$ denote the risk-free compounding factor for the time interval τ to t, given the information I_τ. Then

$$\phi_{\tau t}(F_\tau, F_t) = \frac{u'_t(C_t)}{u'_\tau(C_\tau)} \cdot exp[r_{t-\tau}(t-\tau) \mid I_\tau]$$

The elasticity of the pricing kernel then is, given I_τ,

$$\begin{aligned} \eta(F_t) &= -\frac{\partial \phi_{\tau t}/\partial F_t}{\phi_{\tau t}(F_\tau, F_t)} \cdot F_t = \frac{-u''_t(C_t)}{u'_t(C_t)} \cdot \frac{dC_t}{dF_t} \cdot F_t \\ &= -\frac{u''_t(C_t)}{u'_t(C_t)} \cdot C_t \cdot \frac{dC_t}{dF_t} \cdot \frac{F_t}{C_t} \\ &= R_t(C_t) \cdot \varepsilon_{C_t,F_t} \end{aligned}$$

$R_t(C_t)$ is the representative agent's relative risk aversion and ε_{C_t,F_t} the elasticity of his consumption with respect to the forward price. Hence, the elasticity of the pricing kernel equals the product of the relative risk aversion of the representative agent and the elasticity of his consumption with respect to the forward price. Thus, a sufficient condition for constant elasticity of the pricing kernel is that the representative agent has constant relative risk aversion and constant elasticity of consumption with respect to the forward price. The latter condition holds if aggregate consumption follows a stationary geometric random walk. These cases have been explored in depth by Merton (1971) in a continuous time-setting and by Franke (1984) and Stapleton/Subrahmanyam (1990) in a discrete time-setting. The latter two papers also show that it is very unlikely that the forward price would follow a random walk,unless relative risk aversion is constant and aggregate consumption follows a geometric random walk. Bick (1990) has analyzed a similar problem in a continuous time-setting with consumption at the terminal date T only. Rubinstein (1976) and Brennan (1979) have analyzed a one period model, in which aggregate wealth and the forward price have a bivariate lognormal distribution. They show that the pricing kernel has constant elasticity if and only if the representative agent has constant relative risk aversion.

Since the necessary conditions for constant elasticity of the pricing kernel appear to be very strong, it makes sense to explore the implications of declining elasticity. In general, utility functions with declining relative risk aversion and declining elasticity of aggregate consumption with respect to the forward price are sufficient. Again, these two conditions cannot be viewed in isolation, since even when the aggregate consumption process is exogenously given, the forward price process depends on the utility function. As will be pointed out in the next section, there is some evidence supporting the assumption of declining relative risk aversion.

The existence of a representative agent with time-additive von Neumann-Morgenstern utility is not *required* for a pricing kernel with declining elasticity. Investors can have different utility functions which, in equilibrium, lead to a pricing kernel with declining elasticity. In such a situation, the pricing kernel reflects a sort of weighted average of individual preferences. There is even no need for preferences to be consistent with von Neumann-Morgenstern utility. This follows since no-arbitrage requires only non-satiation with respect to any state-contingent claims.

We have written the elasticity of the pricing kernel $\phi(F_\tau, F_t)$ as $\eta(F_t)$. This implies that $\eta(F_t)$ is independent of F_τ. One might argue that, given F_τ, $\eta(F_t)$ could also depend on F_τ subject to the constraint that $\partial\eta/\partial F_t < 0$. This is not true, however. Clearly, the elasticity of the pricing kernel needs to be path-independent, i.e., independent of the path of the forward price moving from F_τ to F_t. Hence, if the elasticity of the pricing kernel $\phi(F_\tau, F_t)$ depends on F_τ, then for $\theta < \tau$ the elasticity of the pricing kernel $\phi(F_\theta, F_t) = \phi(F_\theta, F_\tau) \cdot \phi(F_\tau, F_t)$ would be path-dependent. In order to see this, consider the elasticities,

$$
\begin{aligned}
\eta(F_t) = \frac{\partial \ln \phi(F_\theta, F_t)}{\partial \ln F_t} &= \frac{\partial \ln \phi(F_\theta, F_\tau)}{\partial \ln F_\tau} \cdot \frac{\partial \ln F_\tau}{\partial \ln F_t} + \frac{\partial \ln \phi(F_\tau, F_t)}{\partial \ln F_t} \\
&= \frac{\partial \ln \phi(F_\tau, F_t)}{\partial \ln F_t}
\end{aligned}
$$

since, given F_τ, $\partial \ln F_\tau / \partial \ln F_t = 0$. Therefore, if the elasticity $\partial \ln \phi(F_\tau, F_t)/\partial \ln F_t$ depends on F_τ, so would the elasticity $\partial \ln \phi(F_\theta, F_t)/\partial \ln F_t$, and hence, the price process would be path-dependent. Therefore, our approach requires that $\eta(F_t)$ is independent of F_τ.

5 Empirical Support for Declining Elasticity

In the analysis so far, declining elasticity of the pricing kernel has been used to generalize the theoretical framework of asset pricing. There is some evidence, however, in favor of this assumption. This evidence can be divided into findings about the asset price process and those about option pricing. We first briefly discuss the evidence about the price process and portfolio choice.

(1) Cross-sectional evidence suggests that poorer households hold a lower proportion of their wealth in risky assets as compared to richer households (Cohn/Lewellen/Lease/Schlarbaum (1975), MacCrimmon/Wehrung (1986), Levy (1994), Oehler (1995, pp.166f)). This suggests that declining relative risk aversion describes the behavior of households better than constant relative risk aversion. Given the close relationship between relative risk aversion and the elasticity of the pricing kernel, declining elasticity seems to be a better assumption than constant elasticity.

(2) Shiller (1981), Grossmann and Shiller (1981) and others document the excess volatility of stock prices, given the volatility of dividends. Proposition 3 says that, given the distribution of the risky asset at date T, the variance of the forward price at any date $t \in (0, T)$ is higher under declining than under constant elasticity. Thus, the observed volatility can be explained better by the declining elasticity of the pricing kernel.

(3) Evidence is accumulating that over long periods returns of stock price indices are mean-reverting and hence negatively autocorrelated (see Poterba and Summers (1988) for example.). Proposition 3 shows that declining elasticity of the pricing kernel implies negative autocorrelation of returns. However, since any kernel with

non-constant elasticity generates negative autocorrelation, the empirical evidence only supports non-constant elasticity, but not necessarily declining elasticity.

We now discuss two empirical findings in option pricing supporting declining elasticity.

(4) Longstaff (1995) finds that options on stock price indices appear to be overpriced relative to the Black/Scholes model[15]. Similarly, it has been noted that the implied volatility of equity index options is often higher than any reasonable estimate of historic volatility (Canina/Figlewski (1993), Christensen/Prabhala (1994)). Our explanation for this is that the elasticity of the pricing kernel is declining instead of constant.

(5) Many empirical studies document that the implied volatility of options depends on the strike price (for an overview see Mayhew (1995)). The shape of the curves relating the implied volatility to the strike price depends on the underlying asset. Options on stock price indices appear to produce a u-shaped curve which is called a volatility smile.

Summarizing, there is substantial empirical support for declining instead of constant elasticity of the pricing kernel.

6 Conclusion

The assumption of a standard geometric Brownian motion for the price of a risky asset is fairly common in financial economics and is used in many valuation models, such as the Black/Scholes model of option pricing. This stochastic process implies that the return process is independent of the price of the asset, and hence, imposes an implicit restriction on the pricing kernel for valuing claims on future payoffs on the asset. In an arbitrage-free market, the pricing kernel is defined such that the forward price of any claim is the expectation of the pricing kernel multiplied by the payoff on the claim. The restriction imposed by the standard geometric Brownian motion is that the elasticity of the pricing kernel with respect to the payoff on the market portfolio is constant.

Although the standard geometric Brownian motion model has been useful in deriving many empirically testable propositions, its characteristics and valuation implications are not always in line with the empirical evidence. Examples of such empirical anomalies include the high volatility of the returns, their autocorrelation and the pricing of contingent claims. The question, therefore, is whether the implicit assumption of constant elasticity can be modified for the resultant models to better fit the data. An alternative proposed and analyzed in this paper is a pricing kernel that exhibits declining elasticity with respect to the payoff on the market portfolio. This model explains a number of empirical anomalies relating to the return generating process and the pricing of contingent claims.

Several other directions of research can be pursued based on the research reported in this paper. First, the additional restrictions on the pricing kernel that would lead to a broader class of stochastic processes for returns than the standard geo-

[15] In recent work, Eom (1996) casts doubt on the conclusion of Longstaff (1995).

metric Brownian motion could be explored. Suitable candidates for such a broader class would include the autoregressive conditional heteroskedasticity (ARCH) models originally proposed by Engle (1982) and their variations, the constant elasticity of variance (CEV) model analyzed by Cox and Ross (1976). These restrictions could be tested directly to assess their empirical validity as has been proposed in the literature on the term structure of interest rates. [16]. Second, the implications of declining elasticity of the pricing kernel for option pricing, such as the "smile" effect, could be explored further. This would provide a better theoretical justification for recent work on fitting binomial trees using observed option prices.[17].

[16]See Constantinides (1992), Turnbull and Milne (1991), and Backus and Zinn (1994), for example.

[17]See Jackwerth and Rubinstein (1995), and Derman and Kani (1994), for example.

Appendix A

Proof of Lemma 1

Consider a set of pricing kernels such that the forward price of the asset is a constant, F_0^*,

$$\{\phi(F_T) \ : \ E[F_T\,\phi(F_T)|I_0] = F_0^*\} \tag{1}$$

By definition of the elasticity of the pricing kernel,

$$\eta(F_T) = -\frac{\partial\phi}{\phi} \Big/ \frac{\partial F_T}{F_T}$$

Consider the two pricing kernels ϕ_1 and ϕ_2 with corresponding elasticities η_1 and η_2 for which

$$\begin{aligned} \frac{\partial\eta_1(F_T)}{\partial F_T} &= 0 \\ \text{and} \qquad \frac{\partial\eta_2(F_T)}{\partial F_T} &< 0\,,\ \forall\, F_T \end{aligned} \tag{2}$$

This implies that

$$\frac{d}{dF_T}\left[\frac{\eta_2}{\eta_1}\right] < 0. \tag{3}$$

First, it is necessary that the two pricing kernels (see Figure 1) intersect at least once. Otherwise, it would be impossible for them to have the same forward price, i.e., $E(\phi_1(F_T)) = E(\phi_2(F_T)) = 1$ would be impossible. Second, the two pricing kernels must intersect more than once,since otherwise the forward price of the risky asset, F_0^*, cannot be the same under both pricing kernels. To see this, suppose that the two pricing kernels intersect only once at $F_T = \hat{F}_T$. Suppose that $\phi_1(F_T) > [<]\phi_2(F_T)$ for $F_T < [>]\hat{F}_T$.Then, consider a claim paying $(F_T - \hat{F}_T)$ at date T. Then, $E[(F_T - \hat{F}_T)\phi_2(F_T)] > E[(F_T - \hat{F}_T)\phi_1(F_T)]$ follows since $(F_T - \hat{F}_T)(\phi_2(F_T) - \phi_1(F_T)) \geq 0\ \forall F_T$. As $E[(F_T - \hat{F}_T)\phi(F_T)] = E[F_T\phi(F_T)] - \hat{F}_T$, the forward price of the risky asset would be higher under pricing kernel $\phi_2(F_T)$ than under $\phi_1(F_T)$. Hence, the forward price can be the same only if the pricing kernels intersect at least twice.

Next, suppose there are three or more intersections of the two pricing kernels. Consider the first three intersections at forward prices F_T^A, F_T^B and F_T^C respectively. Suppose that at F_T^A, ϕ_2 intersects ϕ_1 from above, i.e.,

$$-\frac{\partial\phi_1(F_T^A)}{\partial F_T} < -\frac{\partial\phi_2(F_T^A)}{\partial F_T}$$

Since, at the first intersection,

$$\phi_1(F_T^A) = \phi_2(F_T^A)$$

it follows that

$$\begin{aligned}
\eta_1(F_T^A) &= -\frac{\partial\phi_1(F_T^A)}{\partial F_T}\cdot\frac{F_T^A}{\phi_1(F_T^A)} \\
< \eta_2(F_T^A) &= -\frac{\partial\phi_2(F_T^A)}{\partial F_T}\cdot\frac{F_T^A}{\phi_2(F_T^A)}
\end{aligned} \tag{4}$$

Similarly at F_T^B, ϕ_2 intersects ϕ_1 from below, it follows that

$$\eta_1(F_T^B) > \eta_2(F_T^B) \tag{5}$$

Again, at F_T^C, since ϕ_2 intersects ϕ_1 from above, we must have

$$\eta_1(F_T^C) < \eta_2(F_T^C) \tag{6}$$

However, this would contradict inequality (3). Thus, three or more intersections of the two pricing kernels are not possible. In conclusion, the two pricing kernels must intersect twice and, in order to satisfy equation (2), ϕ_2 must intersect ϕ_1 from above at the first intersection, F_T^A, and from below at the second intersection, F_T^B □

Appendix B
Proof of Proposition 1

Consider European-style call and put options at a strike price K. Define the two intersections of the pricing kernels ϕ_1 and ϕ_2 as F_T^A and F_T^B as illustrated in Figures 1 and 2. Hence, there are three ranges of F_T:

$$\begin{array}{rlll} & F_T < F_T^A & \text{, where} & \phi_2(F_T) > \phi_1(F_T) \\ F_T^A \leq & F_T \leq F_T^B & \text{, where} & \phi_1(F_T) \leq \phi_1(F_T) \\ & F_T > F_T^B & \text{, where} & \phi_2(F_T) > \phi_1(F_T). \end{array} \tag{1}$$

Suppose $K < F_T^A$. It follows that the put option at this strike price is valued higher by the declining elasticity pricing kernel ϕ_2 than the constant elasticity pricing kernel ϕ_1. By put-call parity, the call option at a strike price K is also valued higher by the declining elasticity pricing kernel, ϕ_2.

Suppose $K > F_T^B$. In this case all call options and hence, by put-call parity, all put options are priced higher by pricing kernel ϕ_2 than by ϕ_1.

The more complex case is when

$$F_T^A \leq K \leq F_T^B.$$

Claims that only payoff in the first and third range of F_T are overpriced, by ϕ_2 than by ϕ_1, but claims that payoff only in the middle range are underpriced. Consider a call option at strike price K. The payoff on this option is $g(F_T)$ and its forward price at time 0 is $C_0(K)$. Also, consider a claim with a linear pay-off function $L(F_T)$ where

$$L(F_T) = a + b \cdot F_T \tag{2}$$

where a and b are chosen such that the payoff on this linear claim equals that of the call option at both the crossover points F_T^A and F_T^B, i.e.,

$$\begin{array}{rrcl} & g(F_T^A) & = & L(F_T^A) \\ \text{and} & g(F_T^B) & = & L(F_T^B) \end{array} \tag{3}$$

as shown in Figure 2.

$$\begin{array}{rlll} \text{Hence} & g(F_T) > L(F_T) & \text{for} & F_T < F_T^A, \\ & g(F_T) \leq L(F_T) & \text{for} & F_T^A \leq F_T \leq F_T^B, \\ & g(F_T) > L(F_T & \text{for} & F_T > F_T^B. \end{array} \tag{4}$$

The forward price of the linear payoff $L(F_T)$, $F_0[L(F_T)]$, is the same under both pricing kernels,

$$F_0[L(F_T)|\phi_1] = F_0[L(F_T)|\phi_2] \tag{5}$$

This is true since the linear payoff $L(F_T)$ can be replicated by a combination of the forward contract on the risky asset and a forward contract on the risk free asset. Both of these assets are priced the same by the two pricing kernels, since $E[\phi_1] = E[\phi_2] = 1$ and the forward price of the risky asset is the same, by definition.

The forward price of the call option can be written as the sum of the forward price of the linear payoff and the forward price of the difference between the payoff of the call option and that of the linear payoff

$$C_0(K) = F_0[L(F_T)] + F_0[g(F_T) - L(F_T)]. \tag{6}$$

The first term on the right hand side is the same under the two pricing kernels while the second is not. Consider the second term.

$$F_0[g(F_T) - L(F_T)] = E[\{g(F_T) - L(F_T)\}\phi(F_T)]. \tag{7}$$

Now, consider the difference of this term under pricing kernels $\phi_1(F_T)$ and $\phi_2(F_T)$, which gives us the difference between the forward price of the call option under the two pricing kernels, $C_0(K|\phi_2) - C_0(K|\phi_1)= E[\{g(F_T) - L(F_T)\}\{\phi_2(F_T) - \phi_1(F_T)\}]$. Since the product of the two curved brackets is nonnegative for every F_T (which can be seen from the inequalities in (1) and (4)), it follows that the expectation of this product is positive. Therefore the term in equation (7) is higher under ϕ_2 than under ϕ_1. Hence the forward price of the call option in equation (6) is higher under pricing kernel ϕ_2 than ϕ_1,

$$C_0(K|\phi_2) > C_0(K|\phi_1). \tag{8}$$

By put-call parity, this overpricing result also holds for put options at a strike price K $\square$

Appendix C

Proof of Proposition 2

Consider a European put option with strike price K. Then, the forward price of the put, given the "true" risk-adjusted probability density function $f_2(F_T)$, is defined as P_K,

$$P_K = \int_0^K (K - F_T) f_2(F_T) dF_T. \tag{1}$$

The same forward price is obtained from the Black/Scholes model with the probability density function $f_1(F_T, K)$ which depends on the implied volatility σ_K; this volatility depends on the strike price K. Let P_K^B denote the forward price of the put, derived from the Black/Scholes model with volatility σ_K. It is obtained from equation (1) substituting $f_1(F_T, K)$ for $f_2(F_T)$. Hence

$$P_K = P_K^B \quad ; \forall K. \tag{2}$$

A volatility smile is defined to exist if there exists a strike price $K^\star$ such that

$$\frac{d\sigma_K}{dK} < [=][>] 0 \quad \text{for } K < [=][>] K^\star. \tag{3}$$

First we prove the following lemma.

<u>Lemma 2:</u> *A volatility smile exists if and only if*

$$\int_0^K [f_2(F_T) - f_1(F_T, K)] dF_T < [=][>] 0 \quad \textit{for } K < [=][>] K^\star. \tag{4}$$

<u>Proof:</u> Differentiate equation (2) with respect to K using equation (1).

How is this fx? It seems to be only if

$$\int_0^K f_2(F_T) dF_T = \int_0^K f_1(F_T, K) dF_T + \frac{\partial P_K^B}{\partial \sigma_K} \cdot \frac{\partial \sigma_K}{\partial K}. \tag{5}$$

Since $\partial P_K^B / \partial \sigma_K > 0$, the sign of $\partial \sigma_K / \partial K$ is determined by the difference of the two integrals. Hence, given the definition of a smile, the lemma follows □

Second, we prove the *necessity* of the conditions in proposition 2. Suppose a smile exists. Consider a strike price $K < K^\star$ with K chosen to be small enough so that

$f_2(F_T)$ and $f_1(F_T, K)$ intersect only once in the region $(0, K)$. Let $F_T^*(K)$ be this point of intersection. From equation (2) it follows that

$$\int_0^K (K - F_T)[f_2(F_T) - f_1(F_T, K)]dF_T = 0. \tag{6}$$

Since $(K - F_T)$ is declining in F_T, equation (6) and $\int_0^K [f_2(F_T) - f_1(F_T, K)]dF_T < 0$ can be true only if

$$f_2(F_T) > [=][<] f_1(F_T, K) \text{ for } F_T < [=][>] F_T^*(K). \tag{7}$$

Hence $f_2(F_T) > f_1(F_T, K)$ for $F_T \to 0$ is implied. Now suppose K increases. Then $\partial\sigma_K/\partial K < 0$ implies that $f_1(F_T, K)$ declines for $F_T \to 0$ so that $f_2(F_T) > f_1(F_T, K)$ holds, a fortiori, for $F_T \to 0$. Therefore, this inequality must hold for every $K < K^*$. By an analogous argument, using call options with strike prices $K > K^*$, it follows that $\partial\sigma_K/\partial K > 0$ requires $f_2(F_T) > f_1(F_T, K)$ for $F_T \to \infty$.

Third, we prove *sufficiency* of the conditions in Proposition 2. Assume that these conditions hold. Again, consider a strike price $K < K^*$ such that $f_1(F_T, K)$ and $f_2(F_T)$ intersect only once in the region $(0, K)$. Then $f_2(F_T) > f_1(F_T, K)$ for $F_T \to 0$ and equation (6) imply $\int_0^K [f_2(F_T) - f_1(F_T, K)]dF_T < 0$, and, hence, $\partial\sigma_K/\partial K < 0$, by Lemma 2.
Now suppose K increases subject to $K < K^*$. Then the same argument applies as long as $f_1(F_T, K)$ and $f_2(F_T)$ intersect only once in the region $(0, K)$. Hence, given $f_2(F_T) > f_1(F_T, K)$ for $F_T \to 0$, $\partial\sigma_K/\partial K = 0$ is possible only if $f_2(F_T)$ and $f_1(F_T, K)$ intersect, at least, twice in the region $(0, K)$.
By an analogous argument using call options, $f_2(F_T) > f_1(F_T, K)$ for $F_T \to \infty$ implies for sufficiently high strike prices that $\partial\sigma_K/\partial K > 0$ and that $\partial\sigma_K/\partial K = 0$ is possible only if $f_2(F_T)$ and $f_1(F_T, K)$ intersect, at least, twice in the region (K, ∞). Therefore, in order to have a minimum of σ_K, $f_2(F_T)$ and $f_1(F_T, K)$ must intersect, at least, four times in the region $(0, \infty)$. Since $\partial\sigma_K/\partial K < [>] 0$ for small [high] K, at least one minimum of σ_K must exist. Hence, it remains to be shown that more than one minimum of σ_K cannot exist.
By the assumption of Proposition 2, $f_2(F_T)$ and $f_1(F_T, K)$ intersect, at most, four times. Therefore, it suffices to show that two or more minima of σ_K require more than four intersections. Consider two local minima K^* and K^{**} with $K^{**} > K^*$. Then there must exist three different strike prices K_1, K_2 and K_3 with the same implied volatility σ_{K_1}. Figure 3 illustrates this for the case in which $\sigma_{K^{**}} < \sigma_{K^*}$. In the following we analyse this case, but the analysis would be analogous in the case $\sigma_{K^{**}} \geq \sigma_{K^*}$. At K_1 and K_3, $\partial\sigma_K/\partial K < 0$; at K_2, $\partial\sigma_K/\partial K > 0$. Define $\Delta(F_T) \equiv f_2(F_T) - f_1(F_T, K_1)$. Hence, by Lemma 2,

$$0 > \int_0^{K_1} \Delta(F_T)dF_T < \int_0^{K_2} \Delta(F_T)dF_T > 0 > \int_0^{K_3} \Delta(F_T)dF_T.$$

Note that the condition of Proposition 2 assures $\Delta(F_T) > 0$ for $F_T \to 0$ so that there exists $K_0 \in (0, K_1)$ with $\int_0^{K_0} \Delta(F_T)dF_T > 0$. Therefore, $\Delta(F_T)$ has to change sign at least three times in the region $(0, K_3)$. Since the minimum K^{**} exceeds K_3, $\Delta(F_T)$ must change sign at least twice in the region (K_3, ∞). This follows from the analysis of call options and the condition $f_2(F_T) > f_1(F_T, K)$ for $K > K^{**}$ and $F_T \to \infty$. If there were only one intersection in the range (K_3, ∞), then $\partial\sigma_K/\partial K > 0$ would follow for $K = K_3$. This contradicts the negative slope of the σ_K-function at K_3 in figure 3. Hence, $\Delta(F_T)$ must change sign at least twice in the range (K_3, ∞) and five times in the region $(0, \infty)$. But this contradicts the assumption of at most four intersections of $f_2(F_T)$ and $f_1(F_T, \sigma_{K_1})$. Hence, only one minimum of σ_K can exist □

Appendix D
Declining Elasticity and the Smile

First, we show that declining elasticity of the pricing kernel ϕ_1 generates a smile only if $[\eta_2(F_T) - \eta_1]$ is sufficiently high for $F_T \to 0$. A smile implies inequality (7) in Appendix C. Hence a range of F_T bounded from above, must exist in which

$$\frac{df_2(F_T)}{dF_T} < \frac{\partial f_1(F_T, K)}{\partial F_T}. \tag{1}$$

Consider the first point of intersection, $F_T^\star$, with $f_2(F_T^\star) = f_1(F_T^\star, K)$. Hence inequality (1) implies

$$\frac{d\ln f_2(F_T)}{d\ln F_T} < \frac{\partial \ln f_1(F_T, K)}{\partial \ln F_T} \quad \text{for } F_T = F_T^\star. \tag{2}$$

Each risk-adjusted density is the product of the pricing kernel $\phi(F_T)$ and the density before risk adjustment, $\hat{f}(\ )$. Then it follows from (2) that

$$\begin{aligned} -\eta_2(F_T^\star) + \frac{d\ln \hat{f}_2(F_T^\star)}{d\ln F_T} &< -\eta_1 + \frac{\partial \ln \hat{f}_1(F_T^\star, K)}{\partial \ln F_T} \\ \text{or} \qquad \frac{d\ln \hat{f}_2(F_T^\star)}{d\ln F_T} - \frac{\partial \ln \hat{f}_1(F_T^\star, K)}{\partial \ln F_T} &< \eta_2(F_T^\star) - \eta_1 \end{aligned} \tag{3}$$

Depending on the density functions $\hat{f}_2(F_T)$ and $\hat{f}_1(F_T, K)$, it is possible that the left hand side of inequality (3) is positive; hence the right hand side must be positive, a fortiori. Since inequality (3) must be true for every strike price $K \in (0, K^\star)$ at the first point of intersection, $F_T^\star = F_T^\star(K)$, a smile requires $(\eta_2(F_T^\star(K)) - \eta_1)$ to be sufficiently positive $\forall K \in (0, K^\star)$. Similarly, it can be shown that $(\eta_2(F_T^\star(K)) - \eta_1)$ must be sufficiently negative $\forall K \in (K^\star, \infty)$ with $F_T^\star(K)$ being the last point of intersection of $f_2(F_T)$ and $f_1(F_T, K)$.

Second, we show that the above conditions on elasticities imply a smile. If inequality (3) holds for $F_T^\star(K)$ and $\forall K \in (0, K^\star)$, then $f_2(F_T) > f_1(F_T, K)$ $\forall F_T \in (0, F_T^\star(K))$ and $K \in (0, K^\star))$. Similarly, if inequality (3) with "$<$" being replaced by "$>$" holds at the last point of intersection $\forall K \in (K^\star, \infty)$, then $f_2(F_T) > f_1(F_T, K)$ $\forall F_T \in (F_T^\star(K), \infty)$ and $K \in (K^\star, \infty)$. These implications guarantee the existence of a smile provided that there exist at most four intersections of $f_2(F_T)$ and $f_1(F_T, K)$ $\forall K$ (Proposition 2) $\square$

Appendix E

Proof of Proposition 4

a) Constant and declining elasticities of the pricing kernel are equivalent to constant and declining relative risk aversion in the pricing of claims[18]. Constant relative risk aversion and the lognormality of F_T imply that the expected return $E(F_T/F_{1,t}|F_{1,t})$ is also independent of the state. In the case of declining elasticity, relative risk aversion is smaller, the higher $E(F_T|F_{1,t})$ is, thus implying a lower expected rate of return. Hence, the rate of return in the high states (high $F_{1,t}$) is relatively lower and the rate of return in the low states is relatively higher for the declining elasticity pricing kernel, compared to the constant elasticity case. Therefore, the forward price at time t, $F_{2,t}$, for the declining elasticity case is relatively higher than $F_{1,t}$ in the high states and relatively lower in the low states. Since relative risk aversion is monotonically declining, it follows that $F_{2,t}/F_{1,t}$ is monotonically increasing in $F_{1,t}$, $\forall t \in (0,T)$.

b) Given the same initial price F_0^*, it must be that the forward prices under the two pricing kernels do not dominate each other. Hence, given that the elasticity of ϕ_2 is monotonically declining, there can be only one "crossover point" for the forward prices of the two pricing kernels as a function of the state of nature. Hence, for low values of $F_{1,t}$, $F_{2,t} < F_{1,t}$ and for high values of $F_{1,t}$, $F_{2,t} > F_{1,t}$. In other words, there is a $F_{1,t}^*$, such that $F_{1,t} = F_{2,t} = F_{1,t}^*$, and the result b) follows.

c) From a) and b), it follows that

$$F_{2,t} = F_{1,t} + E[F_{2,t} - F_{1,t}] + \epsilon \tag{1}$$

where $E(\epsilon) = 0$ and $\text{cov}(\epsilon, F_{1,t}) > 0$ since $F_{2,t}$ gets larger relative to $F_{1,t}$ as $F_{1,t}$ increases. Hence,

$$\sigma^2(F_{2,t}) = \sigma^2(F_{1,t}) + \sigma^2(\epsilon) + 2\text{cov}(\epsilon, F_{1,t}) > \sigma^2(F_{1,t}). \tag{2}$$

d) For the constant elasticity pricing kernel, the autocorrelation of returns is zero, since the process is generated by a standard geometric Brownian motion. Now, consider dates $0, t_1, T$ and the returns F_{t_1}/F_0 and F_T/F_{t_1}. If the return in the period $[0, t_1]$ is lower [higher] under non-constant elasticity, then the conditional expected return in the period $[t_1, T]$ must be higher [lower] implying negative autocorrelation. Second, we split the period $[t_1, T]$ into subperiods $[t_1, t_2]$ and $[t_2, T]$. By the same argument as before, given some state at t_1, the returns F_{t_2}/F_{t_1} and F_T/F_{t_2} must be negatively autocorrelated under non-constant elasticity. Similarly, the period $[t_2, T]$ can be split sequentially into arbitrarily many subperiods so that, by induction, negative autocorrelation of returns is obtained for any number of subperiods $\square$

[18]Intuitively, one can think of the relative risk aversion of the representative agent, although this is strictly not necessary.

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