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Who Buys and Who Sells Options: The Role and Pricing of Options in an Economy with Background Risk[1]

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Abstract

In this paper, we derive an equilibrium in which some investors buy call/put options on the market portfolio while others sell them. Also, some investors supply and others demand forward contracts. Since investors are assumed to have similar risk-averse preferences, the demand for these contracts is not explained by differences in the shape of utility functions. Rather, it is the degree to which agents face other, non-hedgeable, background risks that determines their risk-taking behavior in the model. We show that investors with low or no background risk have a concave sharing rule, i.e., they sell options on the market portfolio, whereas investors with high background risk have a convex sharing rule and buy these options. A general increase in background risk in the economy reduces the forward price of the market portfolio. Furthermore, the prices of put options rise and the prices of call options fall. Investors without background risk then react by choosing a sharing rule with higher slope and concavity.

1 INTRODUCTION

The spectacular growth in the use of derivatives to manage risks has been one of the most significant recent developments in the financial markets. In particular, the use of options to hedge against changes in foreign exchange rates, interest rates, equity market prices and commodity prices, is now widespread. In addition, there is increasing interest in real options, such as the option to exploit natural resources, and their role in the theory of investment. Further, many insurance contracts can be thought of as put options. In contrast to the widespread use and importance of options as well as the vast academic and practitioner literature on option pricing, research on explaining the motivation for the use of options is quite sparse. Some explanations have been provided including, for example, the existence of differential transactions costs, heterogeneous expectations and differences in preferences across market participants. However, there has been very little detailed analysis of the reasons why option-like instruments are employed by diverse market participants.

In this paper, we provide a new explanation for option supply and demand: the existence of non-hedgeable background risks. The explanation which we provide is that agents face non-hedgeable, independent, background risks. These risks which could, for example, be associated with labor income or holdings of non-marketable assets, are assumed to be non-insurable. Our model assumes, therefore, that markets are incomplete. Agents faced with such background risks respond by demanding insurance in the form of options on the *marketable* risks.

Furthermore, our model may be able to explain otherwise puzzling behavior. For example, a familiar case is the use of options by corporations that hedge foreign-exchange exposure. A corporation that plans to sell a foreign currency at a future date will often buy a put option on the currency from a counter-party such as a bank. The three standard explanations - heterogeneous expectations, differences in preferences, and transactions costs - for such a transaction are less than plausible, on closer examination. It is difficult to believe that the expectations of industrial corporations consistently differ from those of banks regarding future foreign exchange rates. Also, there is no reason to believe that the shareholders of banking firms have fundamentally different utility functions from those of individual corporations. Further, large organizations, whether they are banks or industrial corporations, are likely to face rather similar transactions costs.

Our alternative explanation relates more to the risk profiles of the two parties. The industrial corporation is likely to face many non-hedgeable risks, such as the risks in the product market. In contrast, the banking firm is exposed mainly to hedgeable market value risks such as those associated with foreign exchange rates and interest rates, or those that can be diversified away to a large extent. In the language of this paper, the industrial corporation faces significant non-hedgeable background risk, whereas the banking firm does not. In our model, we show that the banking firm will tend to sell options and the industrial corporation will tend to buy options.

We assume an economy where agents inherit a portfolio of state-contingent claims on the market portfolio. There is a perfect and complete market for state-contingent claims on this portfolio. All agents in the economy have hyperbolic absolute risk aversion [HARA] utility for wealth at the end of a single time-period. This assumption allows us to compare optimal sharing rules in the presence of background risk with the *linear* sharing rules that exist in an economy with HARA utility and no background risks. The sharing rule tends to be convex for those agents who face high background risk and concave for those who do not. Thus, the non-linearity in our model is attributable to differential background risks. A convex or concave sharing rule can be obtained by buying or selling options, whereas a linear sharing rule involves only the use of spot or forward contracts.

The effect of an independent background risk with a non-positive mean is then analyzed within a comparative statics framework. An increase in the size of the background risk effectively makes an individual agent more averse to market risks. Given the prices of state-contingent claims, the agent's reaction is to demand more claims on states in which the outcome of the market portfolio is low, financing these purchases with sales of claims on states in which the outcome is high. If many agents react similarly in the face of an increase of their background risk, prices of state-contingent claims will change. Equilibrium prices of put options on the market portfolio rise and those of call options fall. Also the forward price of the market portfolio falls.

Next, we analyze the effect of an increase in aggregate background risk on the optimal sharing rules of agents. We show that an agent who happens to face no background risk reacts to the increase in aggregate background risk by choosing a sharing rule with a higher slope and concavity, i.e., by buying fewer claims in the low states and more claims in the high states. In other words, this agent "takes the other side" in response to the increased demand for insurance from agents with background risk.

The organization of the paper is as follows: In section 2, we review the relevant literature on the impact of background risk. In section 3, we assume that a perfect, complete (forward) market exists for state contingent claims on the market portfolio. We define the agent's utility maximization problem in the presence of background risk and illustrate the properties of the precautionary premium given the assumption of HARA preferences. In section 4, we show that, in this economy, the presence of background risk modifies the well-known linear sharing rule.[1] In equilibrium, every agent holds the risk-free asset, the market portfolio and a portfolio of state-contingent claims akin to options on the market portfolio. Agents with high background risk buy these options, whereas those with low background risk sell them. In section 5, we consider the effect of an increase in background risk on the pricing of claims on the market portfolio. We show that an increase in background risk increases the risk aversion of the pricing kernel, reducing the forward price of the market portfolio, increasing

[1] See Cass and Stiglitz (1970) and Rubinstein (1974).

the forward price of put options on the market portfolio, and decreasing the forward price of call options. We also show that an increase in background risk causes the forward prices all *all* options to increase more than does an increase in risk aversion, holding the forward price of the market portfolio constant. In section 6, we analyze the optimal sharing rules of agents in response to an increase in aggregate background risk. In section 7, we conclude with a discussion of these results and their empirical implications.

2 THE PREVIOUS WORK ON BACKGROUND RISK

It has been increasingly recognized in the literature that an agent's behavior towards a marketable risk can be affected by the presence of a second, independent, background risk. Nachman (1982), Kihlstrom et. al. (1981) and Ross (1981) discuss the extent to which the original conclusions of Pratt (1964) have to be modified when a background risk is considered. Recent work by Kimball (1993) shows that if agents are standard risk averse, i.e., they have positive and declining coefficients of risk aversion and prudence, then the derived risk aversion [Nachman (1982)] of the agent will increase with background risk.[2] Further work by Gollier and Pratt (1993), extending results of Pratt and Zeckhauser (1987), shows the effect of the introduction of background risk on weak proper risk aversion, a less stringent condition than standard risk aversion. In this paper, we concentrate on the HARA-class of utility functions, which is a special case of standard risk aversion. This restriction allows us to derive specific results regarding the demand for risky claims by agents in the economy.

In deriving the optimal sharing rules in the presence of non-hedgeable risk, we draw also on the work of Kimball (1990). In particular, we use his concept of the precautionary premium. Further, in the special case of the HARA-class of functions considered in this paper, specific statements can be made about the precautionary premium. This allows us, in turn, to specify the optimal sharing rule and identify the role of hedging with forward contracts and options.[3]

In a recent paper Weil (1992) considers the effect of background risk on the equity premium. He shows that standard risk aversion implies an increase in the equity premium. In this paper, we derive related results for option prices. The above work on background risk has also been applied to the analysis of optimal insurance. Papers by Doherty and Schlesinger (1983a, 1983b) and Eeckhoudt and Kimball (1992) regarding the optimal deductible and the coinsurance rate show that agents expand the coverage of risks in the presence of background risk. Since insurance contracts can be modeled in terms of options,

[2]The coefficient of risk aversion is defined as the negative of the ratio of the second to the first derivative of the utility function. The coefficient of prudence is defined as the negative of the ratio of the third to the second derivative of the utility function.

[3]It should be noted that Briys, Crouhy and Schlesinger (1993) and Briys and Schlesinger (1990) have also previously employed the precautionary premium in the context of hedging.

our results for the demand for options can also be interpreted in terms of the demand for insurance. Finally, there is the related, but distinct work of Leland (1980) and Brennan and Solanki (1981) in portfolio insurance.[4] These papers investigate differences across the utility functions of agents such that they buy or sell options on the market portfolio. They show that agents will demand portfolio insurance if their risk tolerance relative to that of the representative agent increases with the return on the market portfolio. Our analysis is linked to this previous work in the sense that background risk provides a rationale for utility functions to exhibit the properties found to be necessary by Leland. In our economy differences in the risk-taking behavior of agents arise even though the agents have similar utility functions.

3 BACKGROUND RISK, THE DEMAND FOR RISKY ASSETS, AND THE PRECAUTIONARY PREMIUM

We assume a two-date economy where the dates are indexed 0 and 1. There are I agents, $i = 1, 2, \ldots, I$, in the economy. At time 1, X is the risky payoff on the market portfolio. We assume a complete market for claims on the market portfolio so that each agent can buy state-contingent claims on X.[5] In particular, as in Leland (1980), the agent chooses a payoff function, i.e. a sharing rule, which we denote as $g_i(X)$. This function relates the agent's payoff from holding state-contingent claims on the market portfolio to the aggregate payoff, X.

In addition to the investment in the marketable state-contingent claims, the agent also faces a non-insurable background risk. This risk has a non-positive mean and is independent of the market portfolio payoff, X. We denote this background risk as a time 1 measurable random variable, $\sigma_i \varepsilon_i$, where ε_i is a random variable with non-positive mean and unit variance. σ_i is a constant measuring the size of the background risk. The agent's total income at time 1 is, therefore,

$$y_i = g_i(X) + \sigma_i \varepsilon_i \tag{1}$$

The background risk is unavoidable and cannot be traded. The agent can only take this risk into account in designing an optimal portfolio of claims on X. Hence, we investigate the effect of the background risk, $\sigma_i \varepsilon_i$, on the optimal sharing rule $g_i(X)$.

[4] Gennotte and Leland (1990) and Brennan and Schwartz (1989) investigate a related issue: the effect of portfolio insurance on the stock market during the crash of 1987. However, their emphasis is on market liquidity and the effect of hedging in an equilibrium with option-based strategies.

[5] We are concerned, in this paper, with the effect of non-marketable background risk on agents' portfolio behavior. Standard results from portfolio theory would apply to the choice between various marketable assets, and hence, this simplification does not affect the results here.

The agent solves the following maximization problem:

$$\max_{g_i(X)} E[\nu_i[g_i(X) + \sigma_i \varepsilon_i]] \tag{2}$$

$$\text{s.t.} \quad E\left[[g_i(X) - g_i^0(X)]\phi(X)/(1+r)\right] = 0$$

where $\nu_i(\cdot)$ is the utility function of the agent i. In the budget constraint, $g_i^0(X)$ is the agent i's endowment of claims on the market portfolio payoff X, and $\phi(X)$ is the market pricing kernel, which is initially given exogenously and r is the riskless interest rate, also given exogenously.[6] The first order condition for a maximum in (2) is

$$E[\nu_i'(g_i(X) + \sigma_i \varepsilon_i)|X] = \lambda_i \phi(X), \qquad \forall X \tag{3}$$

where λ_i is the Lagrangian multiplier of the budget constraint.

In order to analyze the impact of background risk on the agent's optimal demand for claims on the market payoff, it is useful to introduce Kimball's concept of the precautionary premium. Kimball (1990) defines a precautionary premium, ψ_i, analogous to the Arrow-Pratt risk premium, except that it applies to the *marginal* utility function rather than the utility function itself. For $y_i \equiv x_i + \sigma_i \varepsilon_i$, he obtains

$$E[\nu_i'(x_i + \sigma_i \varepsilon_i)|x_i] \equiv \nu_i'(x_i - \psi_i) \tag{4}$$

where $\psi_i = \psi_i(x_i, \sigma_i)$. The precautionary premium is a function of the market payoff of the agent and the scale of the background risk. It is the amount of the deduction from x, which makes the marginal utility equal to the conditional expected marginal utility of the agent in the presence of the background risk.[7] Substituting (4) in the first order condition (3) yields, for $x_i = g_i(X)$,

$$\nu_i'(x_i - \psi_i(x_i, \sigma_i)) = \lambda_i \phi(X), \ \forall X. \tag{5}$$

We assume that the utility function $\nu_i(\cdot)$ is of the hyperbolic absolute risk aversion (HARA) form

$$\nu_i(y_i) = \frac{1-\gamma_i}{\gamma_i} \left[\frac{A_i + y_i}{1-\gamma_i}\right]^{\gamma_i} \tag{6}$$

where γ_i and A_i are constants and satisfy the condition that $A_i + y_i > 0$, so that the agent's marginal utility is finite. This would, in turn, imply restrictions on $\sigma_i \varepsilon_i$ and $g_i(X)$, which are assumed to be satisfied.[8] Further, we restrict our analysis to cases where $-\infty \leq \gamma < 1$,

[6] In later sections, we will characterize the pricing kernel within an equilibrium and derive comparative statics results relating to it. In a complete market, $\phi(X)/(1+r)$ is the price of a claim that pays \$1 in state X, divided by the probability of occurrence of the state. $\phi(X)$ is the forward price of the claim, which implies that $E[\phi(X)] = 1$.

[7] The precautionary premium can also be related to the Arrow-Pratt risk premium.

[8] Most commonly-used utility functions such as the quadratic, constant absolute risk aversion and constant proportional risk aversion cases can be obtained as special cases of the HARA family, by choosing particular values of γ_i and A_i. In the case of constant absolute risk aversion, $\gamma_i = -\infty$ and $\nu_i(y_i) = -\exp(A_i y_i)$. With $\gamma_i = 0$, we obtain the generalized logarithmic utility function, $\nu_i(y_i) = \ln(A_i + y_i)$.

i.e. those exhibiting constant or decreasing absolute risk aversion. We choose the HARA-class since it is the only class which implies linear sharing rules for all agents, in the absence of background risk.[9]

From equations (5) and (6) it follows that

$$\nu_i'(x_i - \psi_i(x_i,\sigma_i)) = \left[\frac{A_i + g_i(X) - \psi_i(x_i,\sigma_i)}{1-\gamma_i}\right]^{\gamma_i - 1} = \lambda_i \phi(X) \tag{7}$$

Equation (7) reveals that, given the shape of the market pricing function, $\phi(X)$, the sharing rule $g_i(X)$ depends directly on the precautionary premium $\psi_i(x_i,\sigma_i)$. We, therefore, begin by analyzing the effect of the x_i and σ_i on the precautionary premium.

For fairly general utility functions, a number of properties of the precautionary premium, ψ_i, have been established in the literature. Most of these follow from the analogy between the risk premium, π_i, defined on the utility function, and the precautionary premium, ψ_i, defined on the marginal utility function. From the analysis of Pratt-Arrow, π_i is positive and decreasing in x_i, if the coefficient of absolute risk aversion, $a_i(y_i) = -\nu_i''(y_i)/\nu_i'(y_i)$ is positive and decreasing in y_i. Similarly, ψ_i is positive and decreases with x_i, if the coefficient of the absolute prudence, defined as $\eta_i(y_i) = -\nu_i'''(y_i)/\nu_i''(y_i)$ is positive and decreases with y_i (see Kimball, 1990). The correspondence can be taken further. For *small* risks with a zero-mean, the risk premium [precautionary premium] is equal to one-half the product of the coefficient of absolute risk aversion [absolute prudence] and the variance of the payoff on the small risk. For larger risks, higher absolute risk aversion [prudence] implies a higher risk premium [precautionary premium].

Since, for the HARA-class of utility functions, the coefficient of absolute prudence is strictly proportional to the coefficient of absolute risk aversion, $\gamma_i < 1$ implies also positive decreasing absolute prudence and hence, standard risk aversion as defined in Kimball (1993). We now establish the following results regarding the shape of the $\psi_i(x_i,\sigma_i)$ function:

Lemma 1: *In the presence of background risk, if $\nu_i(y_i)$ is of the HARA family with $-\infty < \gamma_i < 1$,*

$$\begin{aligned} \psi_i &> 0, \\ \frac{\partial \psi_i}{\partial x_i} &< 0, \\ \frac{\partial^2 \psi_i}{\partial x_i^2} &> 0, \end{aligned}$$

For $\gamma_i = -\infty$ (exponential utility), $\psi_i > 0$ and $\partial\psi_i/\partial x_i = 0$.

Proof: See Appendix A. □

[9] See Cass and Stiglitz (1970) and Rubinstein (1974)

The significance of Lemma 1 is that it implies that, given a level of background risk, its effect, measured by the precautionary premium, declines and at a decreasing rate in the income from the marketable assets. In other words, the precautionary premium is a positive, decreasing, convex function of the marketable income. The first two statements are implied by positive, decreasing absolute prudence. The exception is the case of the exponential utility function for which the precautionary premium is independent of the marketable income.[10] We are interested also in the effect of the scale of the non-hedgeable background risk, which is indexed by σ_i. Hence, we now establish

Lemma 2: *In the presence of background risk, if $\nu_i(y_i)$ is of the HARA family with $\infty < \gamma_i < 1$,*

$$\begin{aligned}\frac{\partial\psi_i}{\partial\sigma_i} &> 0,\\ \frac{\partial^2\psi_i}{\partial\sigma_i\partial x_i} &< 0,\\ \frac{\partial^3\psi_i}{\partial\sigma_i\partial x_i^2} &> 0.\end{aligned}$$

For $\gamma_i = -\infty$ (exponential utility), $\partial\psi_i/\partial\sigma_i > 0$, but independent of x_i.

Proof: See Appendix A. □

In other words, the increase in the precautionary premium due to an increase in background risk is smaller, the higher the income x; moreover, the convexity of the premium increases as the background risk increases. The first statement in Lemma 2 is implied by positive prudence. The significance of Lemma 2 is that it allows us to compare the effect of background risk on the optimal sharing rules of different agents. Other things being equal, an agent with a higher background risk (larger σ_i) will have a more convex precautionary premium function than one with a lower background risk (σ_i small). Under certain conditions, as we shall see in the next section, this translates into a convex optimal sharing rule in terms of aggregate marketable income.

4 OPTIMAL SHARING RULES

We can now derive the optimal portfolio behavior of agents with different levels of background risk. Then, we can obtain the equilibrium prices of state-contingent claims, which can be studied relative to each other and relative to an economy without background risk. We assume a complete market for state-contingent claims on the market portfolio payoff,

[10] The statements of Lemma 1 regarding $\frac{\partial\psi_i}{\partial x_i}$ and $\frac{\partial^2\psi_i}{\partial x_i^2}$ hold also for a positive mean of the background risk, since the mean has the same effect as adding a constant to x_i.

X. However, agents cannot sell their background risk in the market. Individual agents $i \in [1, 2, \ldots, I]$, choose optimal sharing rules, $g_i(X)$, for claims on X. Agents have HARA class utility functions with $\gamma_i < 1$ and homogeneous expectations regarding the market portfolio payoff. In equilibrium, we require that $g_i(X)$ sums to X over the individual agents. Agents face different levels of background risks, indexed by σ_i, which affect their demands for shares of the market portfolio payoff.[11]

Solving for $g_i(X)$ from equation (7), aggregating over all agents in the economy and imposing the equilibrium market clearing condition $\sum_i g_i(X) = X$, we have

$$X = \sum_i \left[\psi_i(g_i(X), \sigma_i) + [\lambda_i \phi(X)]^{\frac{1}{\gamma_i - 1}} (1 - \gamma_i) - A_i \right], \quad \forall X \tag{8}$$

In principle, (8) can be solved to endogenously determine the market pricing kernel, $\phi(X)$, and then, by substituting back in the individual demand condition, (7), to determine the equilibrium optimal sharing rule, $g_i(X)$, for agent i. However, in general, the resulting expressions for $\phi(X)$ and $g_i(X)$ are complex functions of the parameters γ_i, A_i, and the variables, λ_i, ψ_i for all the agents in the economy.

However, further insight into the portfolio behavior of agents can be gained by assuming that all the agents have the same risk aversion coefficient, γ, but face different levels of background risk, σ_i. This allows us to isolate the effect of the background risk in the portfolio behavior of the agent.[12] If all the agents have the same γ, we can derive a simpler equation for $g_i(X)$. In this case, we have:

Theorem 1: *Assume that agents in the economy have homogeneous expectations and have HARA utility functions with the same γ. Then, assuming that they face different levels of background risk, indexed by σ_i, the optimal sharing rule of agent i is*

$$g_i(X) = A_i^* + \alpha_i X + \alpha_i[\psi_i^*(g_i(X)) - \psi(X)] \tag{9}$$

where

a) $A_i^* = \alpha_i A - A_i$ is the agent's risk free income at time 1, where $A \equiv \sum_i^I A_i$ and

$$\alpha_i = \frac{\lambda_i^{\frac{1}{\gamma-1}}}{\sum_{h=1}^I \lambda_h^{\frac{1}{\gamma-1}}}, \qquad \sum_{i=1}^I \alpha_i = 1,$$

b) $\alpha_i X$ is the agent's linear share of the market portfolio payoff,

c) $\alpha_i[\psi_i^*(g_i(X)) - \psi(X)]$ is the agent's payoff from contingent claims, where

[11] As a special case of the model, where $\sigma_i = 0$ for all i, we have the case explored by Leland (1980).

[12] Leland (1980) focuses on the other case, where there is no background risk, but agents differ in terms of their risk aversion coefficients. He shows that the sharing rule of agent i is convex if and only if γ_i is less than the γ of the representative agent, assuming HARA preferences.

$\psi_i^* = \frac{\psi_i(g_i(X),\sigma_i)}{\alpha_i}$ *and*

$$\psi(X) = \sum_{i=1}^{I} \psi_i(g_i(X)).$$

<u>Proof</u>: Solving (8) for ϕ(X) in the special case where $\gamma_i = \gamma, \forall i$, and substituting in (7) yields (9). □

Theorem 1 does not provide an explicit solution for $g_i(X)$, the sharing rule, since $\psi_i^*(g_i(X))$ and $\psi(X)$ themselves depend on the sharing rule. However, it permits us to separate the demand of the agent for claims on X into three elements. The first two provide a linear share of the market portfolio payoff. If there were no background risk for all agents in the economy, the third element would be zero and the individual agent would have a linear sharing rule (as in Rubinstein (1974)). Note that the linear share represented by the first two elements can be achieved by arranging forward contracts on the market portfolio, or, equivalently, by aggregate borrowing/lending and investment in shares in the market portfolio. The non-linear element is provided by the third term in equation (9). This is non-linear because we know that the precautionary premium ψ_i is a convex function of $g_i(X)$ (Lemma 1). However, in equilibrium, it is the relative convexity of $\psi_i^* = \psi_i/\alpha_i$ compared to the aggregate ψ of all agents in the market that determines the convexity (or concavity) of the sharing rule. Since the third element in the sharing rule is non-linear, it must be achieved by the agent buying or selling option-like contingent claims on the market portfolio. However, whether an individual agent buys or sells such claims depends upon $\psi_i^* = \psi_i/\alpha_i$ compared to the aggregate ψ.[13]

In order to evaluate the sharing rule for a particular agent and to ask whether that

[13] One might conjecture that under appropriate conditions there exists an agent with a linear sharing rule. This is very doubtful, however. The following example shows a situation in which such an agent cannot exist. Suppose there exist three agents. Agent 1 has no background risk. The other two agents have *small* background risks so that

$$\psi_i(x_i, \sigma_i) = 1/2\eta_i(x_i)\sigma_i^2; \qquad i = 2, 3.$$

Now suppose that agent 2 has a linear sharing rule. Then

$$\psi_2(x_2, \sigma_2) = \alpha_2(\psi_2(x_2, \sigma_2) + \psi_3(x_3, \sigma_3))$$

follows from his sharing rule, or

$$\psi_2(x_2, \sigma_2)(1 - \alpha_2) = \psi_3(x_3, \sigma_3)).$$

For small risks it follows

$$\eta_2(x_2)\sigma_2^2(1 - \alpha_2) = \eta_3(x_3)\sigma_3^2.$$

In the HARA-case, this yields

$$\frac{\sigma_2^2(1 - \alpha_2)}{A_2 + x_2} = \frac{\sigma_3^2}{A_3 + x_3}$$

so that x_3 is linear in x_2. Hence linearity of $x_2 = g_2(X)$ implies linearity of $x_3 = g_3(X)$. But then agent 1 must also have a linear sharing rule in equilibrium which contradicts corollary 1.1. Therefore, in this example, a representative agent, i.e. an agent with a linear sharing rule, cannot exist.

agent is, for example, a buyer or seller of options, we need to investigate the convexity of the pricing function $\phi(X)$. For that purpose, we investigate the shape of $\phi(X)^{\frac{1}{\gamma-1}}$, as a function of X.[14] Differentiating equation (7) and aggregating over the agents in the market, we find

$$\left[\frac{\partial\phi(X)^{\frac{1}{\gamma-1}}}{\partial X}\right]^{-1} = (1-\gamma)\sum_i \lambda_i^{\frac{1}{\gamma-1}}\left[1-\frac{\partial\psi_i}{\partial g_i(X)}\right]^{-1} \tag{10}$$

From this equation, it follows that $\partial\phi(X)/\partial X < 0$. This result confirms our intuition that contingent claims on states where X is low are relatively expensive. This conclusion is confirmed in the presence of the non-hedgeable risks. From equation (8) and $\frac{\partial\phi(X)}{\partial X} < 0$, it follows that $\frac{dg_i(X)}{dX} > 0$. Therefore, differentiating (10) with respect to X and applying $\partial^2\psi_i/\partial g_i(X)^2 > 0$ (Lemma 1) we find that $\phi(X)^{\frac{1}{\gamma-1}}$ is a strictly concave function. Now, from the aggregate equation (8), it follows immediately that $\psi(X) = \sum_i \psi_i(X)$ is strictly convex.

Background risk changes $\phi(X)^{1/(\gamma-1)}$ from a linear function of X to a concave function. Therefore, an agent without background risk reacts to this concavity by selling claims in states where X is low or X is high and by buying claims in the other states. This implies a concave sharing rule:

Corollary 1.1: *Suppose that there is an agent who has no background risk in an economy where other agents face background risk. The sharing rule of this agent is strictly concave.*

Proof: Since the agent has no background risk, this follows by placing $\psi_i^* = 0$ in equation (9). Since $\psi(X)$ is convex, as has been shown above, $-\alpha_i\psi(X)$ is concave and the optimal sharing rule for this agent is concave. □

In order to obtain a concave sharing rule, the agent has to sell call and put options at different strike prices. Strictly speaking, options with infinitely many strike prices would be required to exactly construct the desired sharing rule. The essential point is that although the agent may also take positions in linear claims such as forward contracts, options are also required to produce the desired sharing rule.

This is also true of agents with positive background risk who would have a non-linear demand for claims on the market portfolio. This non-linear element is the difference between two convex functions, $\psi_i^*(X)$ and $\psi(X)$. It is difficult to be precise, therefore, about an agent's sharing rule except to say that it will tend to be convex if his precautionary premium (caused by relatively high σ_i) is more convex than that of the average agent in the market. Those agents with relatively high σ_i will tend to buy claims with convex payoffs and those with relatively low σ_i will tend to sell those claims. This is parallel to Leland's result that

[14] $\phi(X)^{\frac{1}{\gamma-1}}$ is a linear function of $X - \psi$, the aggregate wealth reduced by the aggregate precautionary premium. As ψ is non-linear in X, $\phi(X)^{\frac{1}{\gamma-1}}$ is not linear in X, given the background risk.

those agents with relatively low γ_i will tend to buy convex claims. Hence, those agents will have to buy put and/or call options. Therefore, background risk can explain why some agents buy and others sell portfolio insurance.

Next, we can relate our result in Theorem 1 directly to the literature on sharing rules where a two-fund separation is established. Two-fund separation refers to the agent buying a portfolio of riskless securities and a share of a portfolio of risky assets.[15] Theorem 1 indicates that the existence of background risk destroys the two-fund separation property. It is not possible to generalize the result to three-fund separation since the third "fund" varies across agents. To see this note that agents' holdings in the third "fund" net out to zero and hence have the nature of "side-bets". These side bets are similar, however, for those agents with "similar" $\psi_i(g_i(X), \sigma_i)$. In the following corollary, we define "similar" in a precise manner and obtain a three-fund separation result.

Corollary 1.2: *Consider a class of agents $\hat{I}$, defined by the set $\{i \in \hat{I} : \psi_i((g_i(X), \sigma_i)) = q_i \psi_{\hat{I}}(X)$ for $q_i > 0$ $\}$ where $\psi_{\hat{I}}(X)$ is the precautionary premium for the class. A three-fund separation theorem holds in equilibrium for this class of agents.*

Proof: Using Theorem 1 for this case, the optimal sharing rule is

$$g_i(X) = A_i^* + \alpha_i[X - \psi(X)] + q_i\psi_{\hat{I}}(X), i \in \hat{I}. \qquad \square$$

The agent buys a risk-free asset, a share α_i of the market portfolio adjusted for the aggregate precautionary premium, and thirdly, a share q_i of a fund with a non-linear payoff, $\psi_{\hat{I}}(X)$. Note, however, that the condition $\psi_i(g_i(X), \sigma_i) = q_i\psi_{\hat{I}}(X)$ is rather strong. It holds if $A_i + g_i(X) = q_i(A_{\hat{I}} + g_{\hat{I}}(X))$ and $\sigma_i = q_i\sigma_{\hat{I}}$, which implies strict proportionality between q_i and α_i.

It should be noted that all the results of this section are also valid if some agents have background risk with a positive mean. This follows since the size of the background risk is constant and the mean is a constant added to x_i. Thus, the demand and supply of options due to background risk does not depend on the mean of the background risk. In the following sections we have to assume, however, that the means of background risk are non-positive.

5 THE EFFECT OF A CHANGE IN AGGREGATE BACKGROUND RISK ON THE PRICING OF CLAIMS

In this section, we consider the effect of a *general* increase in the background risk of individual agents in the economy. Using certain simplifying assumptions, we show the effect, first

[15] See, for example, Cass and Stiglitz (1970) and Rubinstein (1974).

on the pricing kernel and then on the forward price of the market portfolio and the forward price of options on the market portfolio. As we have seen above, aggregation is difficult to obtain over agents with different HARA utility functions. To clarify the argument, we again make the assumption that the risk aversion parameter, γ, is the same for all agents. In this case, equation (8) can be written in the form

$$\psi(X,\sigma) + \lambda(1-\gamma)[\phi(X)]^{\frac{1}{\gamma-1}} - A = X, \quad \forall X \tag{11}$$

where

$$\begin{aligned} \psi(X,\sigma) &= \sum_i^I \psi_i(g_i(X),\sigma_i) \\ A &= \sum_i^I A_i \\ \lambda &= \sum_i^I (\lambda_i)^{\frac{1}{\gamma-1}} \end{aligned}$$

and σ represents the level of the aggregate background risk.[16] The pricing kernel is, therefore,

$$\phi(X) = \left[\frac{1}{\lambda}\frac{A + X - \psi(X,\sigma)}{1-\gamma}\right]^{\gamma-1} \tag{12}$$

In order to characterize the effects of an increase in background risk on equilibrium prices, it is necessary to make assumptions regarding aggregation. In particular, the aggregate shadow price of the budget constraint, and the aggregate precautionary premium, need to be characterized in a manner similar to the behavior of these variables at the level of the individual agent. Hence, we now assume that the results of Lemma 2 with respect to changes in the background risk hold in aggregate, i.e. $\partial\psi/\partial\sigma > 0$, $\partial^2\psi/\partial\sigma\partial X < 0$, $\partial^3\psi/\partial X^2\partial\sigma > 0$. These mild assumptions regarding the aggregation properties are required to exclude possible complex feedback effects of prices on the composition of agents' portfolios.

These assumptions assure that the aggregate precautionary premium behaves in a manner similar to that of individual agents. We can now establish the following properties of $\phi(X)$, defining $\phi_1(X)$ and $\phi_2(X)$ respectively as the pricing kernels with low and high levels of background risk:

<u>Lemma 3</u>: *Given that the economy satisfies the aggregation property, the pricing kernel $\phi(X)$ has the property*

$$\frac{\partial^2\phi}{\partial X\partial\sigma} < 0 \tag{13}$$

[16]Note that σ is defined implicitly by the aggregate property. It is the background risk that yields the aggregate precautionary premium given the aggregate marginal utility function in (11).

Proof: The proof is in two steps. First, consider $\left(\frac{A+X-\psi}{1-\gamma}\right)^{\gamma-1} \equiv f(X,\sigma)$. Then we have $\partial^2 f/\partial X \partial\sigma < 0$ as

$$\frac{\partial^2 f}{\partial X \partial \sigma} = f_X^{-2}(-\frac{\partial \psi}{\partial \sigma})(1-\frac{\partial \psi}{\partial X}) - f_X^{-1}\frac{\partial^2 \psi}{\partial X \partial \sigma}$$

where

$$f_X^{-1} = \frac{\gamma-1}{A+X-\psi}\, f, \qquad f_X^{-2} = \frac{(\gamma-1)(\gamma-2)}{(A+X-\psi)^2}\, f$$

Next, from the above results and $\phi(X) \equiv f(X,\sigma))/E(f(X,\sigma))$, it follows that $\phi_1(X)$ and $\phi_2(X)$ intersect once and $\phi_2(X)$ has the steeper slope, i.e. $\partial^2\phi(X)/\partial\sigma\partial X < 0$. □

The effect of σ on the pricing kernel in Lemma 3 is illustrated in Figure 1, where the pricing kernels with low and high levels of background risk are shown by $\phi_1\phi_1$ and $\phi_2\phi_2$ respectively. Clearly, since the two curves intersect only once, the implication of the diagram is that contingent claims paying off in low states $X < X^*$ will be priced more highly in the higher background risk economy. Conversely, claims on high payoff states will decline in price. Before drawing conclusions regarding the prices of particular securities, we first derive some further properties of $\phi(X)$.

It is useful now to define the absolute and relative risk aversion for this economy. Since $\phi(X)$ is proportional to the "marginal utility" of this economy, we define the coefficient of absolute risk aversion of the pricing kernel as

$$z(X) = -\frac{\partial\phi/\partial X}{\phi(X)} \tag{14}$$

and the coefficient of relative risk aversion of the pricing kernel as

$$r(X) = -\frac{X\,\partial\phi/\partial X}{\phi(X)} \tag{15}$$

Differentiating (12) we find, for the absolute risk aversion in this case,

$$z(X) = -\frac{\partial\phi/\partial X}{\phi(X)} = (1-\gamma)\frac{1-\partial\psi/\partial X}{A+X-\psi(X)} \tag{16}$$

with an analogous expression holding for $r(X)$. Lemma 4 follows immediately.

Lemma 4: *In an economy composed of agents with HARA preferences and a common risk aversion parameter γ, the coefficients of absolute and relative risk aversion of the pricing kernel are increasing in background risk i.e.*

$$\frac{\partial z(X)}{\partial\sigma} > 0, \qquad \frac{\partial r(X)}{\partial\sigma} > 0$$

Proof: From Lemma 2, and the assumption that changes in background risk have the same impact on the individual and the aggregate precautionary premium, it follows that $\partial\psi/\partial\sigma > 0$, and $\partial^2\psi/\partial X\partial\sigma < 0$. Hence, the numerator of (16) increases with σ. The denominator decreases, since $A + X - \psi(X) > 0$. Hence, $\partial z(X)/\partial\sigma > 0$ and, by a similar argument, $\partial r(X)/\partial\sigma > 0$. □

Lemma 4 is analogous to the classical risk aversion results along the lines of Pratt (1964) for HARA utility functions. The coefficient of absolute risk aversion of the pricing kernel, $z(X)$, is similar to that of the utility function. Hence, there is an analogy between the behavior of the pricing kernel and the utility function.

We are now in a position to analyze the effect of background risk on the value of various contingent claims on the market portfolio payoff. First, consider a forward contract to buy the market portfolio payoff, X. The forward price is the agreed price which makes the forward contract a zero-value contract. Defining this forward price as $F(X)$ we have

$$0 = E[(X - F(X))\phi(X)] \tag{17}$$

or simply

$$F(X) = E[X\phi(X)] \tag{18}$$

Options on the market portfolio payoff are defined in an analogous manner. The forward prices of call and put options on the market portfolio payoff at a strike price K are as follows:

$$C(K) = E[max(X - K, 0)\phi(X)] \tag{19}$$

and

$$P(K) = E[max(K - X, 0)\phi(X)] \tag{20}$$

We can derive the following comparative statics properties of these prices for an increase in the background risk.

Theorem 2:[17] *Given that the economy satisfies the aggregation property, an increase in background risk has the following effects:*

a) The forward price of the market portfolio payoff declines, i.e.

$$\partial F(X)/\partial\sigma < 0$$

b) The forward price of a call option at strike price K declines, i.e.

$$\partial C(K)/\partial\sigma < 0, \forall K$$

[17] The same results can also be derived from a weaker assumption than HARA utility. It suffices that an increase in an agent's background risk makes him/her more averse to marketable risk. If this is true for every agent and a mild aggregation property holds, then the same result can be obtained for more general preferences.

c) The forward price of a put option at strike price K increases, i.e.

$$\partial P(K)/\partial\sigma > 0, \forall K$$

<u>Proof</u>: Consider an increase in the background risk represented in Figure 1. It follows immediately from Lemma 4 that $\partial F(X)/\partial\sigma < 0$, since, from Lemma 4, risk aversion increases with the background risk.

Now consider the value of call and put options on X. The values of all claims to the right of the cross-over point, X^*, decline as background risk increases. Hence, the forward price of a call option at a strike price $K \geq X^*$ declines with an increase in background risk. For $K < X^*$, we can write the forward price of the call option as

$$C(K) = E[max(X-K,0)\phi(X)I_{\{K<X<X^*\}}] + E[max(X-K,0)\phi(X)I_{\{X\geq X^*\}}] \tag{21}$$

where $I_{\{\cdot\}}$ is the indicator function.

Consider an option whose payoff is equal to the payoff of the call option reduced, in every state, by an amount $X^* - K$, the option payoff at X^*, the crossover point between the two pricing kernels. The value of this option is given by

$$\begin{aligned} C(K, X^*-K) &= E[max(X-X^*, -X^*+K)\phi(X)I_{\{K<X<X^*\}}] \\ &\quad E[max(X-X^*, -X^*+K)\phi(X)I_{\{X\geq X^*\}}] \\ &= C(K) - (X^*-K) \end{aligned} \tag{22}$$

Since a constant amount $X^* - K$ is subtracted from the price of the call option to obtain the price of the modified contract, we can write

$$\frac{\partial C(K)}{\partial\sigma} = \frac{\partial C(K, X^*-K)}{\partial\sigma} \tag{23}$$

Now consider $C(K)$ under the two pricing kernels $\phi_1\phi_1$ and $\phi_2\phi_2$ in Figure 1. Its payoff function has two areas. The area to the right of the crossover point, which has a positive payoff, declines in price under pricing kernel $\phi_2\phi_2$ relative to $\phi_1\phi_1$, since the former is lower in this region. The other area to the left of the crossover point has a negative payoff and hence, also declines in price under the pricing kernel $\phi_2\phi_2$ relative to $\phi_1\phi_1$, since the former is higher in this region.

Hence, the forward prices of all call options decline with a rise in background risk. A similar argument can be used to show that the prices of all put options increase with a rise in background risk. □

The effect of an increase in background risk is to reduce the prices of claims in states with relatively high payoff of the market portfolio and increase the prices of claims in states with relatively low payoff of the market portfolio. Furthermore, the "average" price of claims represented by the forward price of the market portfolio also declines. It is apparent from Figure 1 that call options with a high strike price (i.e. to the right of the crossover point X*) get cheaper when the background risk increases. However, this is also true for *all* call options regardless of the strike price. This is intuitively reasonable when one notices that the forward price of a call with a zero strike price changes by the same amount as the forward price of the market portfolio, since these two contracts differ only by a constant in their payoffs. Similarly, the figure shows that a put option with a low strike price becomes more expensive. This is also true of all other put options. Again, the forward price of a put option with a very high strike price changes by the same amount as that of the short position in the market portfolio, since their payoffs differ only by a constant.

Theorem 2 has important testable implications. It allows us to clearly separate the effects of an increase in aggregate background risk from those of an increase in the risk of the market portfolio payoff X. An increase in the latter risk should lower the forward price of the market portfolio because of risk aversion, but raise the forward prices of call options with high strike prices, since for these options the insurance value increases while the intrinsic value is zero anyway. In contrast, an increase in aggregate background risk lowers the forward prices of all these options. Therefore, a situation in which the forward prices of all puts and of calls on the market portfolio with high strike prices increase, indicates a rise in the risk of the market portfolio payoff X. Conversely, a situation in which the forward prices of all puts go up and those of all calls go down, indicates an increase in background risk. Hence this approach allows to distinguish empirically between increases in background risk and increases in the risk of the market portfolio payoff.

How can we distinguish the effects on option prices of an increase in background risk from those that result from a general increase in risk aversion? One way to do this is to compare the effects on option prices of an increase in background risk with those that follow from an increase in risk aversion, where both changes have the *same* effect on the forward price of the market portfolio. The result is shown in the following Theorem 4. It considers the *relative* increase in put and call option prices in the two cases. In the following theorem we consider firstly, an increase in background risk, and secondly, an increase in risk aversion where both changes have the *same* effect on the forward price of the market portfolio. We establish:

Theorem 3: *Assume an economy which satisfies the aggregation property. Suppose initially that there is no background risk in this economy. Consider now the effect of introducing background risk with $\sigma > 0$, with no change in risk aversion, and alternatively an increase in risk aversion caused by a change in γ from γ_0 to γ_1, where these two changes reduce the forward price of the market portfolio by the same amount. Then, the forward prices of all call and put options are higher in the case where background risk increases than in the case where risk aversion increases, i.e.*

$$\textit{If } F(X|\sigma > 0, \gamma = \gamma_0) \quad = \quad F(X|\sigma = 0, \gamma = \gamma_1), \textit{ then}$$

$$
\begin{aligned}
C(K|\sigma > 0, \gamma = \gamma_0) &> C(K|\sigma = 0, \gamma = \gamma_1), \ \forall K, \\
P(K|\sigma > 0, \gamma = \gamma_0) &> P(K|\sigma = 0, \gamma = \gamma_1), \ \forall K.
\end{aligned}
$$
[18]

The notation $C(K|\sigma,\gamma)$ refers to the forward price of a European call option at strike price K given that background risk is at level σ and the risk aversion parameter is at level γ. Similar notation is adopted for put options and for the market portfolio.

Proof:

With background risk, $\sigma > 0$, the pricing kernel is denoted by $\phi_1(X)$. Given the background risk, the pricing kernel and the forward price are given by the following equations

$$\phi_1(X) = \left[\frac{1}{\lambda_1}\frac{A + X - \psi(X)}{1-\gamma_1}\right]^{\gamma_1 - 1} \tag{24}$$

$$F(X|\sigma > 0, \gamma = \gamma_1) = E[X\phi_1(X)] \tag{25}$$

where $E[\phi_1] = 1$, and λ_1 is the shadow price of the budget constraint, taking background risk into account.

Without background risk, $\sigma = 0$, and

$$\phi_2(X) = \left[\frac{1}{\lambda_2}\frac{A + X}{1-\gamma_2}\right]^{\gamma_2 - 1} \tag{26}$$

$$F(X|\sigma = 0, \gamma = \gamma_2) = E[X\phi_2(X)] \tag{27}$$

where λ_2 is again the shadow price of the budget constraint. It follows almost immediately that the risk aversion coefficient $1-\gamma_2 > 1-\gamma_1$.[19] To see this, note that if $\gamma_1 = \gamma_2$, then by Lemma 4, $z(X|\sigma > 0, \gamma = \gamma_1) > z(X|\sigma = 0, \gamma = \gamma_2)$, where the $z(\cdot)$'s are the coefficients of absolute risk aversion in the two cases. This implies that $F(X|\sigma > 0, \gamma = \gamma_1) < F(X|\sigma = 0, \gamma = \gamma_2)$, which contradicts the assumption of the Theorem. If $\gamma_1 < \gamma_2$, the same result obtains, *a fortiori.* □

We now compare the background risk pricing kernel, $\phi_1(X)$ and the kernel $\phi_2(X)$. The following Lemma confirms that the shape of the functions is as shown in Figure 2. The curves must intersect twice at X^1 and X^2.

Lemma 5: *The pricing kernel with background risk $\phi_1(X)$, and the pricing kernel with no background risk and increased risk aversion $\phi_2(X)$, both of which yield the same forward price, intersect twice with*

$$
\begin{aligned}
\phi_1(X) &> \phi_2(X) \quad \text{for } X < X^1, \\
\phi_1(X) &< \phi_2(X) \quad \text{for } X^1 < X < X^2, \\
\phi_1(X) &> \phi_2(X) \quad \text{for } X^2 < X.
\end{aligned} \tag{28}
$$

[18] We consider only non-trivial options such that $\text{Prob}(X < K) > 0$ and $\text{Prob}(X > K) > 0$ for all options, so that there is a non-zero probability of the option finishing out-of-the-money.

[19] Weil (1992) proves this formally for the more general case where $\nu''' > 0$.

Proof: a) First, since the forward price of the riskless asset with a face value of 1 is unity, $E(\phi_1(X)) = E(\phi_2(X)) = 1$. In other words, any pricing kernel must yield a unit forward price for the riskless asset with a face value of one. This restriction, together with $\phi_1(X) \neq \phi_2(X)$, implies that the pricing functionals intersect at least once.

b) Second, note that put-call-parity is violated if there exists only one intersection. Suppose that X^1 is the only intersection point with $\phi_1(X) > \phi_2(X)$, for $X < X^1$, and $\phi_1(X) < \phi_2(X)$ for $X > X^1$. Then, consider the forward price of a call with a strike price X^1. This asset must be overpriced by $\phi_2(X)$ compared to its price using $\phi_1(X)$. However, a put with the same strike price, X^1, must be underpriced by $\phi_2(X)$ relative to $\phi_1(X)$ which contradicts put-call-parity. Hence, just one intersection of $\phi_1(X)$ and $\phi_2(X)$ is not possible.

c) Third, in Appendix B, we show that more than two intersections cannot exist. □

It follows from Lemma 5 that the two pricing kernels are as shown in Figure 2.[20] Given the shape of the pricing kernels in Figure 2, where the background risk kernel and the no background risk, increased risk aversion kernel intersect twice, it is interesting to note[21] that $\phi_1 > \phi_2$ both below X^1 and above X^2. It follows that the price of a put option at a strike price of X^1 or a call option at a strike price of X^2, is higher for *both* options in the case of background risk. However, we can now establish a more general result. If a call option at a strike price X^2 has a lower price using ϕ_2 and a put option at a strike price X^1 also has a lower price using ϕ_2, then *all* put and call options have lower prices using ϕ_2 compared to ϕ_1. The remainder of the proof is given in Appendix C.□

The surprising aspect of this result is that *all* put and call prices are higher in the background risk economy than in the economy with increased risk aversion. It mirrors the fact that the convexity of the pricing kernel is greater in this case. The importance of the result is that it allows us to distinguish the effects of an increase in background risk on option prices relative to those of a similar increase in risk aversion. Suppose we observe a given reduction in the forward price of the market portfolio that could be due either to an increase in the background risk of agents or to an increase in risk aversion. In the former case, both forward prices of puts and calls will be higher. Another way of interpreting Theorem 3 is that forward prices of all put and call options *relative* to the forward price of the market portfolio are higher when background risk increases than when risk aversion increases. The theorem therefore adds empirical content to the previous result in Theorem 2.

[20] The proof of Lemma 5 uses certain properties of the HARA function. Therefore, it is doubtful whether a similar result could be obtained under weaker assumptions.

[21] This is analogous to the result that an agent with no background risk sells claims below X^1 and above X^2 and buys in between.

6 THE EFFECT OF A CHANGE IN AGGREGATE BACKGROUND RISK ON THE OPTIMAL SHARING RULE

In the previous section, we investigated the effect of an increase in aggregate background risk on the forward prices of claims in different states. Since the prices of claims in low states rise and those of claims in high states fall, an investor *without* background risk would be motivated to sell more claims (or buy fewer claims) on low states and buy more claims on the high states. This does not necessarily imply that the agent's sharing rule becomes more concave, in part because the market value of the endowment is also affected, which has a feedback effect on the sharing rule. In order to sort out the "income effect" from the "substitution effect," we need to formally analyze the effect of an increase in aggregate background risk on the agents' portfolio choice.

We will assume that the same aggregation conditions that were assumed in the previous section hold, i.e. $\partial\psi/\partial\sigma > 0$, $\partial^2\psi/\partial\sigma\partial X < 0$, $\partial^3\psi/\partial x^2\partial\sigma > 0$. These assumptions were motivated by reasoning that what is true at the level of the individual agent also holds at the aggregate level. In addition, we assume that the shadow price of the budget constraint for an agent without background risk, say investor $\bar{\imath}$, increases less than for the average agent in the economy. This can be justified by the following consideration.

Evaluating the expectation of equation (5), we have

$$E[\nu_i'(x_i - \psi_i(x_i, \sigma_i))] = \lambda_i, \ \forall i.$$

If σ_i increases, then $\psi_i(x_i, \sigma_i)$ increases, too. Hence, the marginal utility increases. If the sharing rule $x_i = g_i(X)$ were to stay the same, then λ_i would increase. Now consider an investor $\bar{\imath}$ without background risk. For this investor $\psi_i \equiv 0$. It follows that $\lambda_{\bar{\imath}}$ is unaffected, if his/her sharing rule is unaffected by the increase in aggregate background risk. In other words we assume that

$$\alpha_{\bar{\imath}} \equiv \frac{(\lambda_{\bar{\imath}})^{\frac{1}{\gamma-1}}}{\sum_{i=1}^{I}(\lambda_i)^{\frac{1}{\gamma-1}}}$$

does not decrease, when aggregate background risk increases. We can now derive a result about the sharing rule of an investor without background risk.

Theorem 4: *Consider an investor $\bar{\imath}$ without background risk and assume that $\alpha_{\bar{\imath}}$ does not decrease when aggregate background risk increases. Then an increase in aggregate background risk leads to an increase in the slope and concavity of this agent's sharing rule.*

Proof: The slope of the sharing rule is given from (9) by differentiating with respect to X

$$\frac{\partial g_i(X)}{\partial X} = \alpha_i[1 - \partial\psi(X)/\partial X] > 0$$

Hence, $\partial\alpha_i/\partial\sigma \geq 0$ and $\partial^2\psi(X)/\partial X\partial\sigma < 0$ imply that $\partial^2 g_i(X)/\partial X\partial\sigma > 0$. In other words, the slope of the sharing rule increases with background risk, σ. The increase in concavity follows from $\partial^2\psi(X)/\partial X^2 > 0$ and $\partial^3\psi(X)/\partial X^2\partial\sigma > 0$. □

Theorem 4 shows that an agent without background risk takes more marketable risk by changing to a sharing rule with a higher slope. But the slope increases more in the low states than in the high states. These results follow from two considerations: (1) Since most agents face an increase in background risk, while the agents without background risk do not, the agents with increased background risk prefer to take less marketable risk, and hence, demand a sharing rule with a smaller slope. The latter agents induce the one without background risk to use a sharing rule with higher risk by offering higher prices for claims in the low states and lower prices for claims in the higher states. (2) The agents whose background risk increases not only suffer from this increase more in terms of marginal utility in the low than in the high states, but also the concavity of their precautionary premium increases, *ceteris paribus*. Hence, they would want to balance this effect by a more convex sharing rule. This can only be achieved if the agents without background risk use a more concave sharing rule. In general, an increase in aggregate background risk means that agents whose background risk increases relatively more tend to use sharing rules with a smaller slope and less concavity. Agents whose background risk increases relatively little tend to use sharing rules with a higher slope and higher concavity. Agents with small, unchanged background risk tend to sell more portfolio insurance, while agents with higher, strongly increasing background risk tend to buy more portfolio insurance.

7 CONCLUDING COMMENTS

An increase in the background risk faced by some agents in the economy tends to increase the convexity of their sharing rules and their demand for options. Agents with low background risk tend to increase the concavity of their sharing rules and their supply of options. Background risk also increases the forward prices of put options and reduces the forward prices of call options.

From an empirical perspective, the model allows us to distinguish the effects of an increase in background risk from those associated with an increase in either the volatility of the market portfolio or those associated with an increase in market risk aversion. Suppose we observe an increase in the risk premium on the market portfolio. This could be the result of an increase in background risk, market risk, or risk aversion. If background risk is the cause, then we should observe an increase in put prices and a decrease in call prices. If the risk of the market has increased [by a mean preserving spread], put prices rise, but the prices of out of the money calls will rise, while others will fall. If an increase in risk aversion is the cause, then, although put prices will rise and call prices will fall, all option prices will be smaller than they would be if the cause was an increase in background risk. The model also predicts that agents with relatively high background risk will buy options from those who face low background risk. If we could identify groups of individuals with high background risk, for example people with high job insecurity, we should find that they tend to purchase portfolio insurance. Also, if rich individuals tend to have low background risk, relative to their wealth, these people should tend to be sellers of options. These are predictions of the theory which are potentially verifiable with empirical data.

Appendix A
Properties of the Precautionary Premium for the HARA Class of Preferences with $\gamma < 1$

For the HARA class of preferences, with $\gamma < 1$,

$$\nu(y) = \frac{1-\gamma}{\gamma}\left[\frac{A+y}{1-\gamma}\right]^{\gamma} \tag{29}$$

It follows that

$$\nu'(y) = \left[\frac{A+y}{1-\gamma}\right]^{\gamma-1} > 0 \tag{30}$$

$$\nu''(y) = -\left[\frac{A+y}{1-\gamma}\right]^{\gamma-2} < 0 \tag{31}$$

$$\nu'''(y) = \frac{\gamma-2}{\gamma-1}\left[\frac{A+y}{1-\gamma}\right]^{\gamma-3} > 0 \tag{32}$$

$$a(y) = \left[\frac{A+y}{1-\gamma}\right]^{-1} > 0 \tag{33}$$

$$\eta(y) = \frac{\gamma-2}{\gamma-1}\left[\frac{A+y}{1-\gamma}\right]^{-1} > 0 \tag{34}$$

We can now prove the various statements of Lemmas 1 and 2.

1) Proof that $\psi > 0$.

For the HARA utility function, the marginal utility function ν' is a strictly convex function since $\nu''' > 0$. As a result, we have from Jensen's inequality

$$\begin{aligned}\nu'[x - \psi(x,\sigma)] &\equiv E[\nu'(x+\sigma\varepsilon)|x] \\ &> \nu'[E(x+\sigma\varepsilon)|x] = \nu'(x + \sigma\ E(\varepsilon))\end{aligned} \tag{35}$$

since the risk ε has a non-positive mean. Hence,

$$\psi > -\sigma\ E(\varepsilon) \tag{36}$$

because ν' is a strictly decreasing function of x. □

2) Proof that $\partial\psi/\partial x <$ [=] 0.

We have for a HARA utility function

$$\eta(x) \equiv -\frac{\nu'''}{\nu''} = \frac{\gamma-2}{\gamma-1}a(x) \tag{37}$$

where $a(x)$ is the Arrow-Pratt measure of risk aversion. Hence,

$$\text{sign } \eta(x) = \text{sign } a(x), \qquad \text{sign } \frac{d\eta(x)}{dx} = \text{sign } \frac{da(x)}{dx} \tag{38}$$

It follows from arguments of Pratt (1964) about $a(x)$ that

$$\frac{\partial \psi}{\partial x} < [=]\ 0 \tag{39}$$

where the inequality holds for decreasing absolute risk aversion and the equality holds for exponential utility ($\gamma = -\infty$) for which $a(x)$ is constant. □

3) Proof that $\partial\psi/\partial\sigma > 0$.

By analogy with the arguments of Pratt (1964) and Rothschild and Stiglitz (1970) , since

$$\nu' > 0,\ \nu'' < 0 \implies \partial\pi/\partial\sigma > 0$$

we can write that

$$\nu'' < 0,\ \nu''' > 0 \implies \partial\psi/\partial\sigma > 0 \qquad \square$$

4) Proof that $\partial^2\psi(x,\sigma)/\partial x \partial\sigma < 0$.

For simplicity of notation, we drop the condition "$|x$" in the following equations. Differentiate the definitional equation

$$\nu'[x - \psi(x,\sigma)] \equiv E[\nu'(x + \sigma\varepsilon)] \tag{40}$$

with respect to σ and obtain

$$\frac{\partial\psi(x,\sigma)}{\partial\sigma} = \frac{E[\nu''(x+\sigma\varepsilon)\varepsilon]}{-\nu''[x-\psi(x,\sigma)]} \tag{41}$$

$$= \frac{E[\nu''(x+\sigma\varepsilon)\varepsilon]}{E[-\nu''(x+\sigma\varepsilon)]} \cdot \frac{E[-\nu''(x+\sigma\varepsilon)]}{-\nu''[x-\psi(x,\sigma)]} \tag{42}$$

The second term on the right hand side of equation (42) is positive, given the assumption of risk aversion. Since the left hand side is positive, both fractions on the right hand side of (42) are positive. We now show that both fractions decline in x.

Differentiate the first fraction with respect to x. The differential is negative if and only if

$$E[\nu''(x+\sigma\varepsilon)]E[\nu'''(x+\sigma\varepsilon)\varepsilon] > E[\nu''(x+\sigma\varepsilon)\varepsilon]E[\nu'''(x+\sigma\varepsilon)] \tag{43}$$

which is the same as

$$\frac{E[\nu'''(x+\sigma\varepsilon)\varepsilon]}{E[\nu'''(x+\sigma\varepsilon)]} < \frac{E[-\nu''(x+\sigma\varepsilon)\varepsilon]}{E[-\nu''(x+\sigma\varepsilon)]} \tag{44}$$

since $E[\nu''(x + \sigma\varepsilon)] < 0$ and $E[\nu'''(x + \sigma\varepsilon)] > 0$.

Consider an agent facing the choice between a riskless and a risky asset, where the excess return on the risky asset is equal to $\check{\mu} + \varepsilon$, and $\check{\mu} + E(\varepsilon)$ is the expected excess return of the risky asset over the riskless rate. Let σ denote the optimal dollar investment in the risky asset, given another utility function with marginal utility being equal to $-\nu''(\cdot)$. Then, the optimality condition is that the right hand side of inequality (44) equals $-\check{\mu}$, with x being the riskfree income plus $\sigma\check{\mu}$. For a utility function with higher absolute risk aversion the same fraction would be smaller than $-\check{\mu}$, since the optimal investment in the risky asset would be smaller. As for the HARA class with $\gamma < 1$, $-\nu''''(\cdot)/\nu'''(\cdot) > -\nu'''(\cdot)/\nu''(\cdot) > 0$, inequality (44) holds. This proves that the first fraction on the right hand side of (42) declines in x.

In order to show the same for the second fraction, define

$$\nu''[x - \varphi(x, \sigma)] \equiv E[\nu''(x + \sigma\varepsilon)] \tag{45}$$

where φ is the premium defined by the second derivative of the utility function. [π is the premium defined by the utility function (risk premium) and ψ is the premium defined by the first derivative (precautionary premium)]. Then, the second fraction in (42) can be rewritten as

$$\frac{E[-\nu''(x + \sigma\varepsilon)]}{-\nu''[x - \psi(x, \sigma)]} = \frac{\nu''[x - \varphi(x, \sigma)]}{\nu''[x - \psi(x, \sigma)]} \tag{46}$$

For the HARA class of preferences, the right hand side of (46) can be written as

$$\frac{\nu''[x - \varphi(x, \sigma)]}{\nu''[x - \psi(x, \sigma)]} = \left(\frac{A + x - \varphi(x, \sigma)}{A + x - \psi(x, \sigma)}\right)^{\gamma - 2} \tag{47}$$

Differentiate the right hand side of (47) with respect to x. The differential is negative (since $\gamma < 1$), if

$$(A + x - \varphi)^{-1}\left(1 - \frac{\partial\varphi}{\partial x}\right) > (A + x - \psi)^{-1}\left(1 - \frac{\partial\psi}{\partial x}\right) \tag{48}$$

We substitute for $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\psi}{\partial x}$ by differentiating (40) and (45) to obtain

$$\left[1 - \frac{\partial\psi(x, \sigma)}{\partial x}\right] = \frac{E[\nu''(x + \sigma\varepsilon)]}{\nu''[x - \psi(x, \sigma)]} \tag{49}$$

$$\left[1 - \frac{\partial\varphi(x, \sigma)}{\partial x}\right] = \frac{E[\nu'''(x + \sigma\varepsilon)]}{\nu'''[x - \varphi(x, \sigma)]} \tag{50}$$

We substitute (49) and (50) in (48) to yield

$$\frac{E\left[(A + x + \sigma\varepsilon)^{\gamma - 3}\right]}{[A + x - \varphi]^{\gamma - 2}} > \frac{E\left[(A + x + \sigma\varepsilon)^{\gamma - 2}\right]}{[A + x - \psi]^{\gamma - 1}} \tag{51}$$

Substitute for the denominators in the two sides of the inequality from (40) and (45) and obtain

$$E\left[(A+x+\sigma\varepsilon)^{\gamma-3}\right]E\left[(A+x+\sigma\varepsilon)^{\gamma-1}\right] > \left[E\left\{(A+x+\sigma\varepsilon)^{\gamma-2}\right\}\right]^2 \tag{52}$$

Since

$$(A+x+\sigma\varepsilon)^{\gamma-3}(A+x+\sigma\varepsilon)^{\gamma-1} = \left[(A+x+\sigma\varepsilon)^{\gamma-2}\right]^2 \tag{53}$$

it follows from Cauchy's inequality that (52) holds. Hence $\partial^2\psi/\partial x\partial\sigma < 0$. □

5) Proof that $\partial^2\psi/\partial x^2 > 0$.

From equation (49), it follows that

$$\frac{\partial^2\psi(x,\sigma)}{\partial x^2} > 0 \tag{54}$$

if and only if the right-hand side in equation (49) decreases as x increases. We have already shown this to be true in equations (46) through (53). □

6) Proof that $\partial^3\psi/\partial\sigma\partial x^2 > 0$.

First, note that convexity of ψ approaches 0 as $\sigma \to 0$. Since ψ is convex for any positive value of σ, it follows that convexity increases with σ for small changes from $\sigma = 0$. We now use a monotonicity result to show that convexity increases with σ for any value of σ. We rewrite equation (40) for the HARA class and multiply throughout by $(1/\sigma)^{\gamma-1}$ to obtain

$$\left[\frac{[A+x]}{\sigma} - \frac{\psi(x,\sigma)}{\sigma}\right]^{\gamma-1} = E\left[\left(\frac{[A+x]}{\sigma} + \varepsilon\right)^{\gamma-1}\right] \tag{55}$$

Multiply and divide equation (55) throughout by q, where q>0, to yield

$$\left[\frac{q[A+x]}{q\sigma} - \frac{q\psi(x,\sigma)}{q\sigma}\right]^{\gamma-1} = E\left[\left(\frac{q[A+x]}{q\sigma} + \varepsilon\right)^{\gamma-1}\right] \tag{56}$$

Define x_1 such that

$$q[A+x_0] \equiv A + x_1.$$

Then, using subscript 0 for x in equation (56) yields

$$\left[\frac{[A+x_1]}{q\sigma} - \frac{q\psi(x_0,\sigma)}{q\sigma}\right]^{\gamma-1} = E\left[\left(\frac{q[A+x_0]}{q\sigma} + \varepsilon\right)^{\gamma-1}\right] \tag{57}$$

In words, if σ changes from σ to $q\sigma$ and x changes from x_0 to x_1, then the new precautionary premium $\psi(x_1, q\sigma) = q\psi(x_0,\sigma)$. In order to show that the convexity of ψ grows with σ, suppose that σ is raised from a level arbitrarily close to 0. Then, the convexity of $\psi(x_0,\sigma)$ increases. Hence, the convexity of $\psi(x_1, q\sigma)$ increases by the factor q. As q can be arbitrarily large, the convexity of $\psi(x_1, q\sigma)$ increases monotonically with $q\sigma$. □

Appendix B

Proof that more than two intersections of $\phi_1(X)$ and $\phi_2(X)$ is not possible

Given $\phi_1(X)$ and $\phi_2(X)$, define the risk aversions of the two pricing kernels as

$$z_1(X) = -\frac{\partial\phi_1/\partial X}{\phi_1(X)} \tag{58}$$

$$z_2(X) = -\frac{\partial\phi_2/\partial X}{\phi_2(X)} \tag{59}$$

First, we show that

$$\frac{d}{dX}\left[\frac{z_1(X)}{z_2(X)}\right] < 0$$

To establish this, note that

$$\frac{d}{dX}[z_1(X)/z_2(X)] < 0 \quad \text{iff} \quad \frac{d}{dX}[lnz_1(X) - lnz_2(X)] < 0$$

$$\begin{aligned}
&\frac{d}{dX}[lnz_1(X) - lnz_2(X)] \\
&= \frac{d}{dX}[ln(1-\psi'(X)) - ln(A+X-\psi(X)) + ln(A+X)] \\
&= -\frac{\psi''(X)}{1-\psi'(X)} - \frac{1-\psi'(X)}{A+X-\psi(X)} + \frac{1}{A+X} \\
&= -\frac{\psi''(X)}{1-\psi'(X)} - \frac{z_1(X,\sigma)}{1-\gamma_1} + \frac{z_2(X,0)}{1-\gamma_2} \qquad (60) \\
&< 0
\end{aligned}$$

By assumption, $\psi''(X) > 0$, and hence the first term is negative; by Lemma 4, $z_1(X)$ increases with σ, so that the last two terms together are negative. Hence, the whole expression is negative so that $d[z_1(X)/z_2(X)]/dX < 0$.

Now suppose that there exist at least three points of intersections X^1, X^2, and X^3. Suppose that at X^1, $\phi_1(X)$ intersects $\phi_2(X)$ from above, i.e., $\frac{\partial\phi_1(X)}{\partial X} < \frac{\partial\phi_2(X)}{\partial X}$. Since $\phi_1(X^1) = \phi_2(X^1)$, it follows that

$$z_1(X^1) > z_2(X^1).$$

At X^2, $\phi_1(X)$ intersects $\phi_2(X)$ from below so that it follows $z_1(X^2) < z_2(X^2)$. At X^3 , we must have $z_1(X^3) > z_2(X^3)$ since $\phi_1(X)$ must intersect $\phi_2(X)$ from above. This contradicts $\frac{d}{dX}[\frac{z_1(X)}{z_2(X)}] < 0$. With two intersections, this also implies that at X^1, $\phi_1(X)$ must intersect $\phi_2(X)$ from above and at X^2, $\phi_1(X)$ intersects $\phi_2(X)$ from below. □

Appendix C

Proof of Lemma 5

To establish Lemma 5, consider, for example, the call option with a strike price K shown in Figure 3 and the linear payoff function defined by the line $L(X)$:

$$L(X) = a + bX \tag{61}$$

where a and b are chosen so that $L(X)$ equals the option payoff at both cross-over points X^1 and X^2 i.e. $L(X^1) = C_{X^1}$ and $L(X^2) = C_{X^2}$. C_X denotes the payoff of a call with a strike price K. $L(X)$ is a portfolio of the risk-free payoff and the payoff on a forward contract. Thus, $L(X)$ is priced the same by both pricing kernels. Therefore, it suffices to show that the differential payoff from the call and $L(X)$ is underpriced by ϕ_2. The forward price of the option, using $\phi_1(X)$, is

$$\begin{aligned} C(K,\phi_1) &= E[L(X)\phi_1(X)] + E_1[(C_X - L(X))\phi_1(X)] \\ &+ E_2[(C_X - L(X))\phi_1(X)] + E_3[(C_X - L(X))\phi_1(X)] \end{aligned} \tag{62}$$

where $E_1(\cdot)$ denotes the expectations operator over the interval $X < X^1$, $E_2(\cdot)$ over the interval $X^1 < X < X^2$, and $E_3(\cdot)$ over the interval $X > X^2$. Equation (62) states that the value of the option is the value of the linear payoff $L(X)$ plus the value of the difference between the option payoff and $L(X)$ in each of the three segments. Now, the value in the case of no background risk and higher risk aversion is

$$\begin{aligned} C(K,\phi_2) &= E[L(X)\phi_2(X)] + E_1[(C_X - L(X))\phi_2(X)] \\ &+ E_2[(C_X - L(X))\phi_2(X)] + E_3[(C_X - L(X))\phi_2(X)] \end{aligned} \tag{63}$$

Comparing (63) with (62), the first term is the same in both equations, i.e.

$$E[(a+bX)\phi_1(X)] = E[(a+bX)\phi_2(X)]$$

since

$$E[X\phi_1(X)] = E[X\phi_2(X)]$$

Referring to Figure 3, since $\phi_1(X) > \phi_2(X)$ and $C_X > L(X)$ for the first segment, $X < X^1$,

$$E_1[(C_X - L(X))\phi_1(X)] > E_1[(C_X - L(X))\phi_2(X)]$$

Next, since $\phi_1(X) < \phi_2(X)$ and $C_X < L(X)$ for the second segment, $X^1 < X < X^2$,

$$E_2[(C_X - L(X))\phi_1(X)] > E_2[(C_X - L(X))\phi_2(X)]$$

Finally, in the third segment, since $\phi_1(X) > \phi_2(X)$ and $C_X > L(X)$ for $X > X^2$,

$$E_3[(C_X - L(X))\phi_1(X)] > E_3[(C_X - L(X))\phi_2(X)]$$

It follows, therefore that $C(K,\phi_1) > C(K,\phi_2)$. Also, all puts have higher prices under ϕ_1 using the same argument or by using put-call parity, since the forward price of the asset is the same. □

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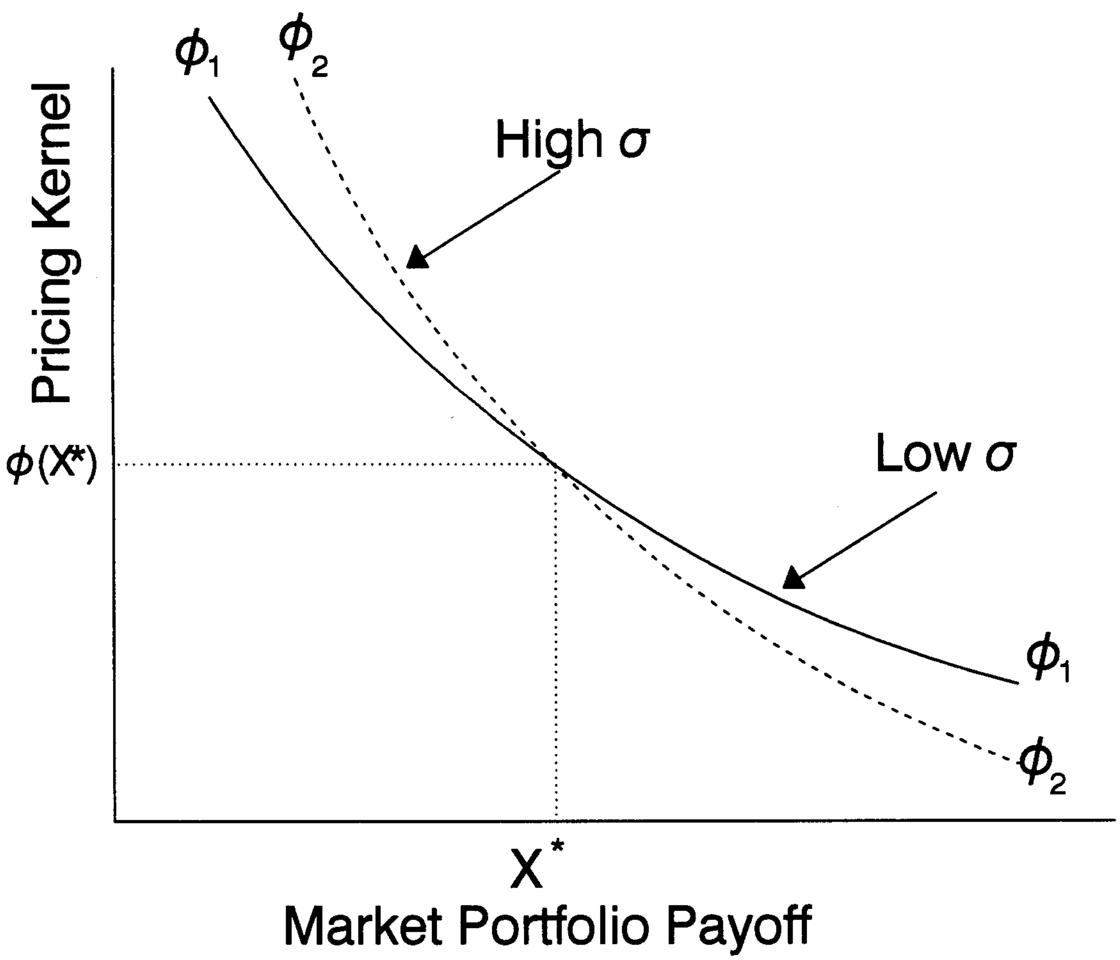
Figure 1

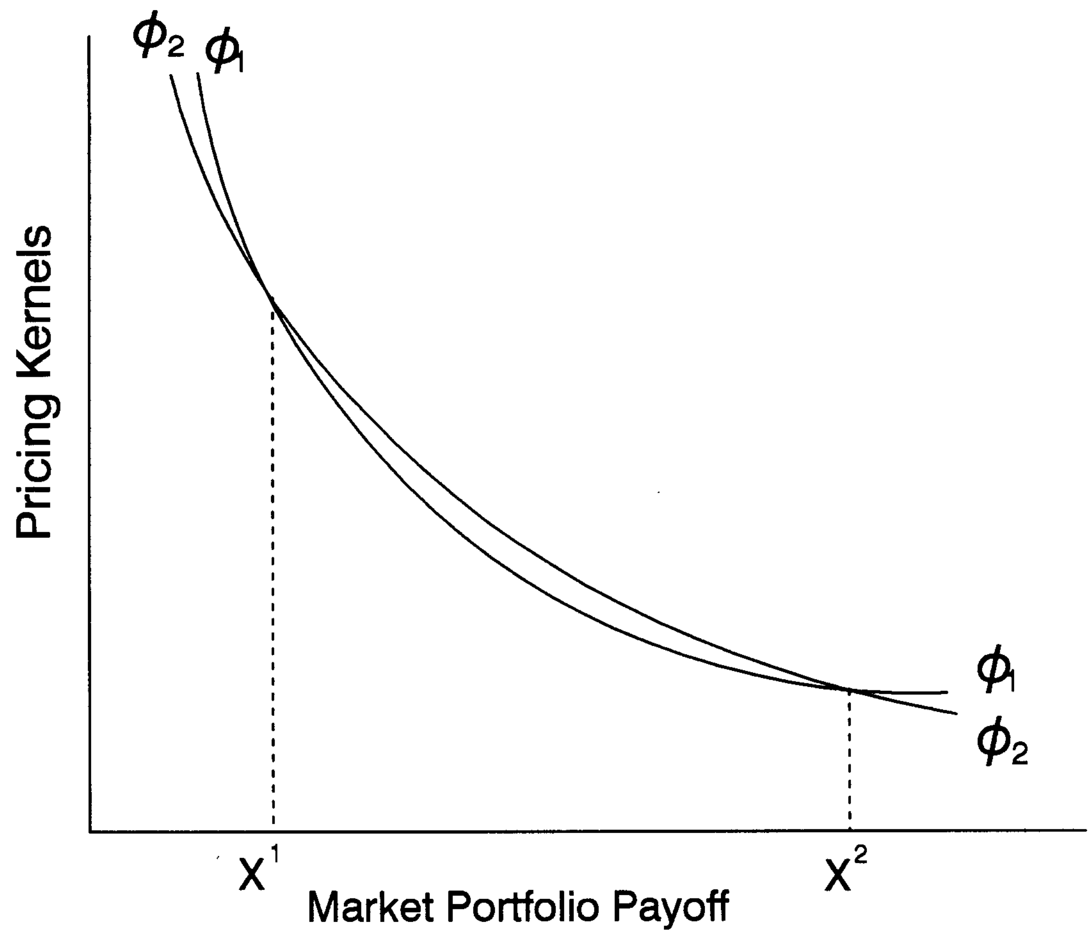
The Effect of an Increase in the Background Risk on the Pricing Kernel



Relationship between the pricing kernel, $\phi(X)$, and the level of the aggregate market payoff, X. The solid line ($\phi_1\phi_1$) represents the relationship between the pricing kernel and aggregate market payoff for low levels of background risk. The dotted line ($\phi_2\phi_2$) represents the same relationship for high levels of background risk.

Figure 2

Pricing Kernels: Background Risk and Increased Risk Aversion



Relationship between the pricing kernel, $\phi(X)$, and the level of aggregate market payoff, X. $\phi_1 \phi_1$ is the pricing kernel with background risk and $\phi_2 \phi_2$ is the pricing kernel without background risk, but with the higher risk aversion. As is evident from the figure,

for $X < X^1$, $\phi_1(X) > \phi_2(X)$

$X > X^2$, $\phi_1(X) > \phi_2(X)$ and

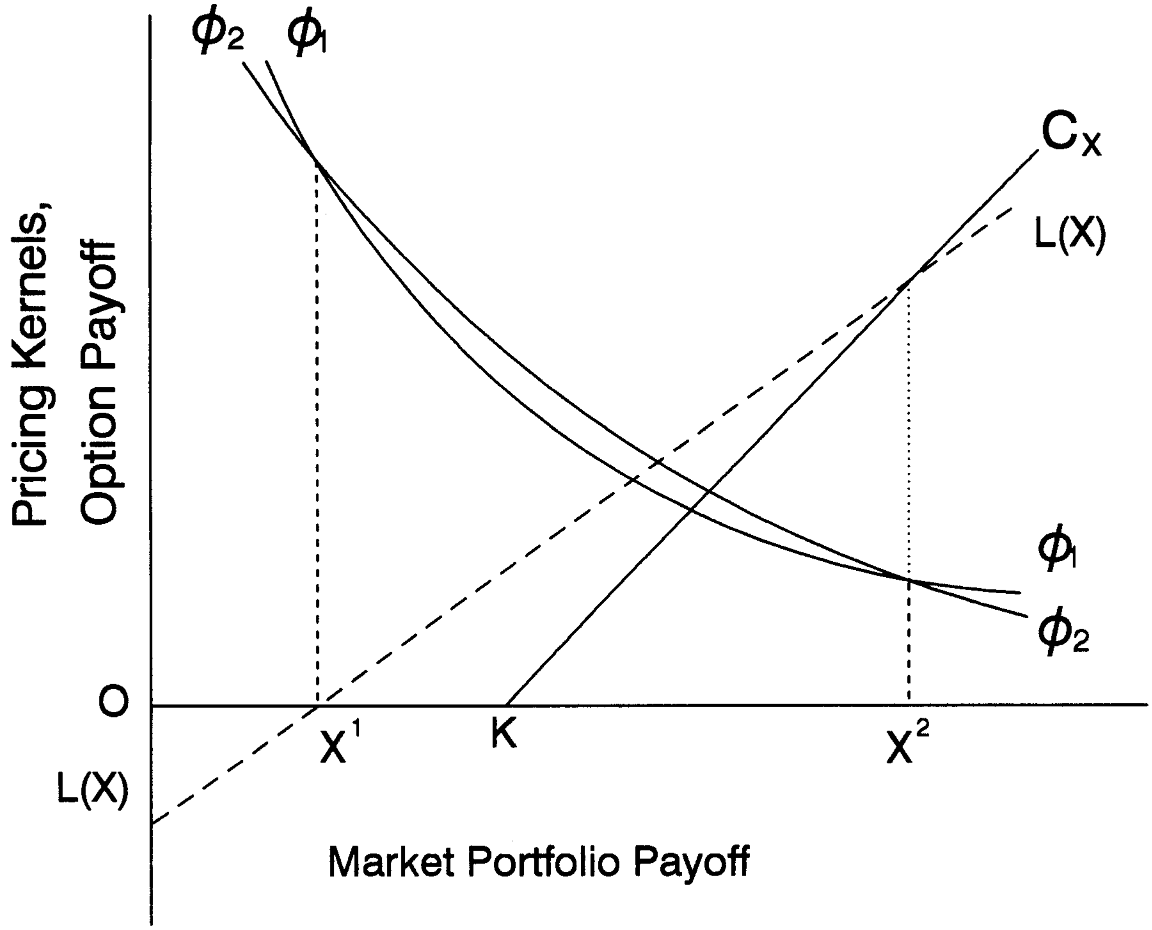
$X^1 < X < X^2$, $\phi_1(X) < \phi_2(X)$; where

X^1 and X^2 are defined such that

at $X = X^1, X^2$, $\phi_1(X) = \phi_2(X)$.

Figure 3

The Value of a Call Option Under the Background Risk and Increased Risk Aversion Pricing Kernels



Relationship between the pricing kernel, $\phi(X)$, and the level of aggregate market payoff, X. ϕ_1 ϕ_1 is the pricing kernel with background risk and ϕ_2 ϕ_2 is the pricing kernel without background risk but with increased risk aversion.

The payoff on a call option at a strike price K is given by the line segments OKC_X.
$L(X) = a + bX$ is a linear payoff such that $L(X^1) = C_{X^1}$ and $L(X^2) = C_{X^2}$