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Alternate Hedge Ratios for Bonds Subject to Credit Risk

To date there is no satisfactory way to measure and control interest rate risk for bonds subject to high levels of credit risk. In addressing this gap, this work develops the survival measure, a new measure of interest rate sensitivity for corporate bonds. The survival measure leads to the development of nine alternate hedge ratios, seven of which are new. Considerable variation in the size of alternate hedge ratios are observed , including some that are consistently larger and others that are consistently smaller than Macaulay based duration hedge ratios now in use. This suggest that improvements in hedging strategies may be available, depending on whether credit risky bonds have a consistently greater (less) response to a change in the level of interest than that suggested by the Macaulay duration based hedge ratio.

In addition, these hedge ratios are evaluated on high credit quality and low credit quality bond portfolios and on well diversified 18 bond portfolios and less diversified 9 bond portfolios. In this way information concerning the role of credit quality and of diversification in hedging performance is examined as well.

Alternate Hedge Ratios for Bonds Subject to Credit Risk

With the advent of the high yield (junk) bond market an important new instrument became available for portfolio managers seeking higher yields by bearing additional risk. However measuring and controlling risk in the high yield bond market is problematic. High yield bonds, like all corporate bonds, are subject to credit, liquidity, and interest rate risk.[1] Where applicable, bonds are also subject to covenant risk (i.e. call risk). Certainly some measures can be taken to control credit risk through use of bond ratings, discriminate analysis (e.g. ZETA analysis) and diversification. Additionally, in a competitive market investors should expect to receive adequate compensation for bearing liquidity and covenant risk. But how to measure and control interest rate risk for bonds subject to significant credit risk? This is the question addressed here.

Current practice is to use a modified version of Macaulay duration as a measure of interest rate risk and to construct hedge ratios. However empirical evidence (Ilmanen, McGuire and Warga 1994, Ilmanen 1992, Toevs and Jacob 1987, and Landes, Stoffels and Seifert 1985) find that while duration based hedging techniques perform as well as more sophisticated regression based techniques, hedging performance becomes poor as credit quality declines.

The first objective is to develop a better summary statistic for measuring interest rate risk for corporate bonds subject to significant credit risk. This is accomplished through use of an alternate bond pricing method developed by Jonkhart (1979). Unlike the usual bond pricing model the Jonkhart model separates the credit risk parameter from the pure interest parameter. These parameters are readily observable or have good empirically measured approximations. Using this approach, we develop two readily measurable interest sensitivity statistics, one based on the bond's price response to a change in the credit risk parameter, and another based on the bond's price response to a change in the pure interest parameter. One of these, the survival interest rate sensitivity measure, is new. These two interest sensitivity measures, along with modified Macaulay duration lead to nine alternate hedge ratios, seven of which are new, that are appropriate for empirical testing on corporate bonds.

A second objective is to evaluate these nine hedge ratios for high credit quality and low credit quality bond portfolios. Considerable variation in the size of hedge ratios are observed, including ratios that are larger and others that are smaller than the ones calculated from modified Macaulay duration. These new hedge ratios may improve hedging strategies if credit risky bonds have a consistently greater (less) response to a change in the level of interest rates than that suggested by the duration based hedge ratio used in practice. Each new ratio, along with modified Macaulay duration and modified Macaulay duration with convexity based hedge ratios, is used to construct hedges for well diversified corporate bond portfolios of controlled levels of credit risk. Measuring the six month percentage gain or loss on the hedged relative to the unhedged position formed on November 1989 provides an indication of their usefulness in measuring and controlling interest rate risk for bond portfolios subject to varying levels of credit risk. Some preliminary evidence suggests that the survival interest rate sensitivity measure developed here can improve hedging performance for low credit quality bonds.

A final objective is to evaluate these hedge ratios for well diversified 18 bond portfolios and less diversified 9 bond portfolios. In this way information concerning the role of diversification in hedging performance is examined as well. These results suggest that diversification plays an important role in reducing the unique aspects of credit risk, particularly for the low credit bond investments.

Section I reviews the literature while Section II develops a new measure for interest rate risk for bonds subject to credit risk. A series of numerical exercises are performed to study behavior of the new interest rate risk measure, along with existing risk measures, under controlled conditions. From this new measure, and existing measures, a total of nine hedge ratios are derived in section III. Each hedge ratio is derived from the objective of minimizing changes in the initial price of a hedge portfolio. An empirical example is then conducted in section IV to evaluate the performance of the hedge ratios. This paper concludes with summary comments and suggestions for further study in Section V.

I Literature Review

Till now research and practice have focused on the use of various measures of duration as a summary statistic for interest rate risk suitable for developing hedge ratios. Kolb and Chaing (1981) first proposed a duration based hedging model for interest rate risk and refinements were contained in Gay and Kolb (1984). Toevs and Jacob (1987) conduct a comparison of alternate hedging strategies and conclude that the duration based hedge is more convenient and performs as well as more sophisticated regression based techniques. Ilmanen (1992) finds that the simplest measure of duration, namely modified Macaulay's duration, explains 80%-90% of the cross-sectional variance in strait (non callable) government bond returns. Empirical results such as these endorse the use of duration based hedge ratios and immunization techniques in controlling the level of interest rate risk in Treasury bonds.[2]

Measuring and controlling interest rate risk for bonds subject to credit risk is another matter. Ilmanen, McGuire and Warga (1994) find that modified Macaulay duration is able to explain nearly 90% of the cross-sectional variation in non-callable Aaa rated bonds, and almost 80% of Aa bonds, but only about 35% of A and Baa bonds. Landes, Stoffels and Seifert (1985) find that Macaulay based duration hedge ratios were able to hedge 78 to 83% of the price variation of diversified, ten bond Aaa to Aa value weighted portfolios but only 60 to 64 % of similar portfolios comprised of Bb and B rated bonds during the calendar years of 1978 and 1979. Evidently, duration is approximately correct for high credit quality bonds, but this approximation becomes progressively poorer as credit quality declines.

Further information is provided by Leyland (1994) who models the price of a corporate bond in a time homogeneous environment. Here corporate bonds continuously "roll over" a fraction *m* of face value such that a constant net amount always remains outstanding. By varying the rate of roll over, bonds of any arbitrary maturity can be modeled. In this framework, the comparable Macaulay duration (effective duration) is less than the traditional Macaulay duration; can even be negative, and can exhibit concavity rather than convexity. He concludes that proper hedging of fixed income securities must consider potential default risks, which Macaulay duration ignores.

It is interesting to note that in the time period covered in the Landes, Stoffels and Seifert (1985) study, interest rates generally rose, leading to overall net losses on high and low credit quality bond hedge portfolios. Their hedge ratio was measured as d_cB_c/d_hB_h where d refers to Macaulay duration, B is price and the subscripts c and h refer to the cash instrument (corporate bond portfolio) and hedging instrument (T-Bond interest rate futures) respectively. This suggests that bonds subject to credit risk tend to have greater sensitivity to changes in interest rates relative to Treasury bonds than measured by their hedge ratio of Macaulay durations. In other words, Macaulay duration is "too small" for corporate bonds and this bias leads to the shorting of too few T-Bond interest rate futures. This results in overall losses as interest rates rise. Further, the hedging strategy was less successful in protecting against losses on the low quality portfolio. This suggests that low quality portfolios are even more interest rate sensitive relative to government bonds than high credit quality bonds.

On the other hand, Leyland (1994) suggests a proper measure of interest rate sensitivity adjusted for default risk would be even less that Macaulay duration. This means that Macaulay duration is "too large" and the resulting hedge ratio would lead to the shorting of too many default risk free bonds. Then the Landes, Stoffels and Seifert (1985) would be explained away as due to the peculiarities of a small sample during a relatively short time interval. In any event this raises the question: Is there a systematic bias, one that increases as credit quality declines, in the use of Macaulay duration for constructing hedge ratios?

II Alternate Interest Rate Risk Measures

Jonkhart (1979) incorporates default risk into the well known unbiased expectations theory of the term structure. In doing so he presents an alternate pricing model for bonds subject to default risk . Under a flat term structure assumption this model is[3]

$$B = \sum_{t=1}^{n} \frac{\{\rho^t\, {}_o r_n + (1-\rho^t)\delta\}(\prod_{j=1}^{t} \rho^{j-1})}{(1+{}_oI_n)^t} + \frac{\prod_{t=1}^{n} \rho^t}{(1+{}_oI_n)^n} \qquad \text{Eq (1)}$$

where ρt = the probability of survival, conditional on no previous default[4]
${}_o r_n$= coupon rate
δ = recovery fraction in the event of default
${}_o I_n$= default risk free (pure) interest rate.

In this formulation, investors are risk neutral in assessing the probability of survival (and of default) to determine the expected cash flows which are discounted at the risk free rate. Unlike the usual bond pricing equation, this formulation separates the credit risk parameters, ρ and δ, from the pure interest parameter, I. This permits us to examine how interest rate changes effects bond prices through changes in credit risk separately from its direct effect on I.

Taking the partial derivative of Eq (1) with respect to ρ, $dB/d\rho$ =,

$$\sum_{t=1}^{n} \frac{\{(t\rho^{t-1}\, {}_o r_n - t\rho^{t-1}\delta)(\rho^{.5(t-1)t}) + .5(t-1)t\rho^{.5(t-1)t-1}\, (\rho^t\, {}_o r_n + (1-\rho^t)\delta\}}{(1+{}_oI_n)^t}$$

$$+ \frac{.5(n+1)n\rho^{.5(n+1)n-1}}{(1+{}_oI_n)^n} \qquad \text{Eq(2)}$$

and taking the partial derivative of Eq(1) with respect to I we obtain

$$dB/di = \sum_{t=1}^{n} \frac{-t[\{\rho^{t}\,{}_{0}r_{n} + (1-\rho^{t})\delta\}(\prod_{j=1}^{t} \rho^{j-1})]}{(1+{}_{0}I_{n})^{t+1}} - \frac{n \prod_{t=1}^{n} \rho}{(1+{}_{0}I_{n})^{n+1}} \qquad \text{Eq(3)}$$

Equation (3) reveals the sensitivity of the bond's price to the pure rate of interest. This measure has been derived by Fooladi, Roberts and Skinner (1996). Equation (2) reveals the sensitivity of a bond's price to a change in the probability of survival[5] .

We would like to model the effect of interest rate changes on both ρ and I. However, to do so we must introduce another equation as we cannot solve for two unknowns with one equation. When introducing another equation, another parameter is inevitably introduced which leads to the need for a third equation and so on. This is a typical problem in Finance and Economics, too many parameters, not enough equations.[6]

To avoid this problem, we model the effect of interest rate changes on corporate bond values as either operating through the pure rate of interest, I or through the survival rate ρ. This is in contrast with Macaulay duration, which implicitly uses some combination of I and ρ as pre-specified in the credit risky yield to maturity. For equation 3 this poses no special problem, since the derivative itself directly measures the corporate bond's price response to a change in the pure rate of interest. The change in price with respect to a change in interest would be $\Delta B \cong (dB/dI \times 1/B) \times \Delta I \times B$, where dB/dI is equation 3. However, how do we measure the bond's price response to a change in interest rates if the primary mechanism is a second parameter, the survival rate ρ ? Equation 2 is a new measure of a bond's price response to a change in the interest rate only if we make an assumption concerning how the pure rate of interest affects the probability of survival.

We choose an assumption that is consistent with current practice. When using the Macaulay duration based hedge ratio, yield spreads are assumed constant.[7] This means that in response to a change in the pure rate of interest, corporate bond yields, Y changes by the same amount. The change in price with respect to a change in interest would be $\Delta B \cong (dB/dY \times 1/B) \times K \times \Delta I \times B$, where $dB/dY \times 1/B$ is modified Macaulay duration and $K = \Delta Y/\Delta I = 1$. For equation 2, the constant yield spread assumption would have the change in ρ in response to a 100 basis point increase in the pure rate of interest result in the same price response as a 100 basis point change in yield Y using the traditional bond pricing equation. The change in price with respect to a change in interest would be $\Delta B \cong (dB/d\rho \times 1/B) \times K \times \Delta I \times B$, where $dB/d\rho$ is equation 2 and $K = \Delta\rho/\Delta I$. K would be empirically measured by finding that $\Delta\rho$ which results in the same price decrease in equation 1 as would a 100 basis point increase in yield Y in the traditional bond price equation. In this way, equation 2 can be used as an interest sensitivity measure and the results are comparable to modified Macaulay duration as the corresponding yield spread assumption is the same.

Comparative static's of Equation (2) reveals that $dB/d\rho > 0$, and $d^2B/d\rho^2 > 0$. In other words, price increases at an increasing rate with ρ. For equation (3), $dB/dI < 0$ and $d^2B/dI^2 > 0$. In other words, price decreases at a decreasing rate with I. To develop more information about the behavior of these derivatives under controlled conditions, a numerical example is now performed.

The base case considers two types of bonds, a high credit bond with a one period probability of survival of .9995 and low credit bond with a probability of survival of .985. Both bonds have a recovery fraction equal to 40%. Altman and Kishore (1996) find that in practice the actual recovery fraction varies greatly with calendar time and credit rating but an overall average in the bond market consistently hovers around 40%.[8] To hold maturity and bond price dimensions constant, both bonds mature in 10 years with a semi-annual pay coupon (6.504% and 17.992% at annual rates) sufficient to price these bonds at par according to equation 1. The pure rate of interest is 6%. The price response of each bond to successive 20 basis point increments in the pure rate of interest on a semi annual basis, both increases and decreases, to a total of a 200 basis point change is calculated. The results are reported in figure 1. A second numerical example examines the price response of each bond to successive 2 basis point increments in the probability of survival on a semi annual basis to a total of a 4 basis point increase and a 30 basis point decrease. These results are reported in figure 2.

[Figures 1 and 2 about here]

In both figures the high credit bond appears to have a substantial amount of convexity. The low credit bond on the other hand, appears to have less convexity. These results suggest that to model the price response to a change in the rate of interest, either through the pure interest rate itself or through its effect on the survival rate, one should consider the second derivative as in both cases convexity may be significant.

To model the price response in the face of convexity, a second order Taylor expansion is employed. In the case of survival convexity, the total price response is approximately,

$$\Delta B \cong (dB/d\rho \times 1/B)(d\rho) + 1/2(d^2B/d\rho^2 \times 1/B)(d\rho)^2 + \varepsilon \qquad \text{Eq(4)}$$

where $d^2B/d\rho^2$ is the second derivative of the price equation and ε is an error term. Taking the second derivative of the price equation we have

$$d^2B/d\rho^2 =$$

$$\sum_{t=1}^{n} \{[(t^2-t)\rho^{t-2}\ {}_or_n - (t^2-t)\rho^{t-2}\delta]\rho^{.5t(t-1)} + .5(t^2-t)\rho^{.5t(t-1)-1}\ (t\rho^{t-1}\ {}_or_n - t\rho^{t-1}\delta) +$$

$$[.5(t^2-t)\{.5(t^2-t)-1\}\rho^{.5t(t-1)-2}]\ (\rho^t\ {}_or_n + (1-\rho^t)\delta) + (t\rho^{t-1}\ {}_or_n - t\rho^{t-1}\delta)[.5(t^2-t)\rho^{.5t(t-1)-1}]\}$$

$$\div (1+{}_oI_n)^t + [(.5(n^2+n)\{.5(n^2+n)-1\})\rho^{.5n(n+1)-2}] \div (1+{}_oI_n)^n \qquad \text{Eq (5)}$$

We employ exactly the same approach to approximating the bond price response in the face of pure interest rate convexity.

The same base case as reported above is employed to determine the adequacy of the second order Taylor expansion in approximating the bond price response in the case of survival and pure interest rate convexity. For comparison purposes modified Macaulay duration and modified Macaulay duration with convexity are also included. Table 1 shows the errors in the price response approximations as calculated from modified Macaulay, modified Macaulay with convexity, pure interest rate, pure interest rate with convexity, survival and survival with convexity for the high credit bond. Each pricing error is calculated by subtracting the actual price from the projected price as determined from use of the first or both terms of the appropriate Taylor expansion (Equation 4). This means that positive differences represent projected prices that are above actual. Actual prices are determined by applying the parameter change to reprice the bond according to Equation (1) or the traditional bond pricing equation in the case of corporate bond yield to maturity changes. Table 2 reports the same information with respect to the low credit bond.

Each table reports the error in response to successive doubling of the critical parameter in question on a semiannual basis from 2 basis points to 264 basis points, both increases and decreases. That is, the survival "duration" and survival convexity columns report errors in response to changes in the survival probability, pure interest rate "duration" and convexity columns report pricing errors in response to changes in the pure rate of interest and modified Macaulay duration and convexity columns report the pricing error in response to changes in the yield.

Of course the high credit bond cannot improve its survival rate above 5 basis points, so only improvements of 4 basis points are shown.

Before a discussion of these results, a word of caution is in order. It is tempting to make judgments on the basis of these results as to which measure is "better". This is erroneous. This exercise yields information about how well we can expect each measure to approximate the price response to a change in the critical parameter in question. Whether one measure is "better" than another will depend on how a corporate bond price actually responds to a change in the interest rate, whether the main response is through a change in the pure rate of interest, the survival rate or some combination, and whether this modeling approach is better than simply using a combination of the pure rate of interest and survival rate as pre-specified in using the corporate bond yield. This can be determined only by empirical work when we examine which hedge ratio derived from these measures can best minimize pricing errors of a hedge portfolio.

[Table 1 about here]

Table 1 reveals that modified Macaulay duration well approximates price changes in response to a change in yield. Only minor improvements at extreme interest rate changes over 200 basis points on an annual basis are obtained by including convexity. The same observations apply to pure interest rate "duration" and convexity. Survival "duration" well approximates price changes in response to a change in the survival rate only for "small" changes in the survival probability. Bear in mind however, a change in the survival rate results in a much stronger price response than a corresponding change in either the pure interest rate or yield. For example, a 4 basis point decline in the survival rate is roughly equivalent to a 32 basis point increase (64 basis points on an annual basis) in the pure interest rate and the yield. A comparison of the errors of pure interest rate and modified Macaulay durations associated with a 32 basis point increase in interest rates with the error associated with survival "duration" for a 4 basis point decline in the survival rate reveals that survival "duration" does approximate the price response about as well as the two interest rate durations.

Table 2 repeats the above observations for a low credit quality bond with one exception. It is noticeable that all measures of interest rate sensitivity model the low credit quality bonds actual price even better than the high credit quality bond of Table 1.

[Table 2 about here]

Altogether Tables 1 and 2 reveal that each of the three measures of price response to a change in interest rates, along with their respective measures of convexity, are viable candidates for empirical consideration.

III Alternate Hedge Ratios

An acid test of a measure of interest rate sensitivity in modeling a bond's price response to a change in interest rates is its' use as a hedge ratio. The measure which develops a hedge ratio that minimizes price differences overall will be the best measure of interest rate sensitivity, demonstrating its superior ability to track the actual price change. Consequently, a derivation of a correct hedge ratio for each variation of the three interest rate sensitivity measures, survival, pure interest and Macaulay, is a vital step.

From these three interest sensitivity measures, a total of nine hedge ratios can be derived. The two existing hedge ratios are the modified Macaulay based hedge and the modified Macaulay based hedge ratio with convexity. The new hedge ratios can be classified as single factor and interaction models. There are four single factor models, the pure interest based hedge ratio, the pure interest hedge ratio with convexity, the survival based hedge ratio, and the survival with convexity based hedge ratio. The three interaction models are the survival and pure interest rate hedge ratio, the survival and pure interest with convexity and the survival and pure interest, both with convexity.[9] With so many new hedge ratios to present, it is best for brevity's sake to present the derivation of the survival hedge ratio, regulating the survival with convexity ratio which represents a slight variation of the below derivation, to appendix 1. The other seven ratios are derived in exactly the same manner.[10]

The objective is to minimize changes in the beginning value of a hedge portfolio.[11] The hedge portfolio is

$$V_h = B_c + NB_h$$

where B_c = the price of the cash instrument
B_h= the price of the hedge instrument
N = the hedge ratio.

Since $\Delta\rho = -K\Delta i + \varepsilon$, $\Delta B_c \cong S_c B_c \Delta\rho$, $\Delta B_h \cong -D_h B_h \Delta i$, then

$$\Delta V_h \cong S_c B_c(-K\Delta i + \varepsilon) + -ND_h B_h \Delta i$$

where $K = \Delta\rho/\Delta i$, S_c is equation 2 (adjusted by dividing by B_c), the survival "duration", and D_h is Macaulay modified duration. Since $E(\varepsilon) = 0$, $E(\Delta i) = 0$ and $E(\Delta i,\varepsilon) = 0$, and using the properties of the expectations operator, the variance of changes in the value of the hedge portfolio is,

$$\mathrm{Var}[\Delta V_h] = [(S_c B_c\text{-}K)^2 + 2S_c B_c\text{-}K\text{-}ND_h B_h + (-ND_h B_h)^2]\ \mathrm{Var}(\Delta i) + (S_c B_c)^2\ \mathrm{Var}(\varepsilon).$$

Taking the first derivative with respect to N, setting the result equal to zero and solving for N we obtain,

$$N = -S_c B_c K / D_h B_h. \qquad \text{Eq (6)}$$

This is the survival based hedge ratio. Table 3 reports all the hedge ratios, each of which were derived in the same manner as above.

[Table 3 about here]

IV Empirical Results

The "best" hedge ratio would be the one that minimizes changes in the initial value of a hedge portfolio. There are nine hedge ratios based on the three competing interest rate sensitivity measures, modified Macaulay, pure interest and survival, or their combination. Once determined, the best hedge ratio would reveal which measure or which combination of measures, is the best summary statistic of interest rate sensitivity.

Each hedge portfolio consists of a long position in a value weighted corporate bond portfolio, the initial value of which is to be hedged, and a short position in a Treasury bond. The Treasury bond that has the closest maturity to the average maturity of the corporate bond portfolio is the hedging instrument.[12] The hedge ratio determines the short position in the Treasury bond. Two hedge portfolios, one each for a portfolio of "AA" rated bonds and for "B" rated bonds, are constructed on November 30, 1989 so the hedge ratios are evaluated for high credit and low credit bonds. To diversify credit risks that are unique to a particular bond issue, each corporate bond portfolio consists of 18 bond issues randomly chosen from eligible issues.

The data used to form portfolios is The University of Wisconsin-Milwaukee's Fixed Income Data Base. This data base consists of monthly information on most publicly traded bonds since 1973. Each issue is identified by cusip number and includes information on issue and maturity date, flat price (noted as quote or matrix based), coupon, accrued interest, bond rating, Macaulay duration and convexity, and call, sinking and put features.

The procedure for corporate bond portfolio formation begins with selection of all bonds with either a "AA" rating or a "B" rating by both Moodys, and Standard and Poors. No distinction is made between bonds with modifiers indicating that a bond was at the upper or lower end of the rating category. Since the above interest rate sensitivity measures do not account for the effect of option like features, all bonds that are putable and immediately callable are eliminated. Unfortunately, it was not possible to select only non-callable bonds since virtually all corporate bonds are callable, so only those bonds with significant call protection were selected. Besides not being immediately callable, the bond must be call protected for three years (representing more than one-third of its remaining life in many cases) or the call price must be five percent over the current price[13].

In addition, empirical criteria were applied to further cull the sample. Bonds which mature within seven years are eliminated since we would like to examine bonds that are likely to exhibit strong price reactions to changes in the general level of interest rates. Matrix priced bonds were eliminated since Sarig and Warga (1989) and Warga (1991) find that use of matrix based prices are liable to lead to serious errors. Finally, bonds whose price quote was less than 50 were eliminated. Altman and Kishore (1996) find that a good average value of bonds that have defaulted is 40, so bonds priced below 50 are likely to be in default, or are just about to do so and should not be rated even B.

The final corporate bond portfolios are formed by random selection from the culled corporate bond sample in the following way. All "AA" rated bonds meeting the above criteria are numbered, as is a separate sample meeting the same criteria but rated "B". For each of these two final samples, the total number of bonds are divided by 18 to determine the whole number to be used as the selection criteria. Then each bond that corresponds to the whole number or its multiple is included in the final portfolio.

However, a final step is necessary since the Fixed Income Data Base does not give details about call status. Each selected bond from the above procedure was checked against Moodys Bond Record to assure bonds met call protection criteria. If they did not, then the bond was replaced by the next bond in sequence that did meet the call protection and all other criteria.

For each bond in the final sample, modified Macaulay duration, pure interest and survival interest rate sensitivity measures, along with the corresponding measures of convexity, are calculated. For the survival and pure interest measures, the recovery rate is assumed to be 40 percent. Altman and Kishore (1996) find that in practice the actual recovery fraction varies greatly with calendar time and credit rating but an overall average in the bond market consistently hovers around 40%.

Exactly what is involved is demonstrated through use of a specific example. Arkansas Best Corp., cusip # 040789AB matures on November 1, 1998. As of November 30, 1989 it is quoted flat at 94, along with accrued interest of $4.9653 so the full price is $95.129 and its full annual bond yield is 15.245%. All of this information is contained in the Fixed Income Data Base and it is used to calculate the semi-annual Macaulay duration of 10.37179 and semi-annual convexity of 68.80217, which agrees with the data base's corresponding annual measurements[14]. The appropriate zero coupon default free interest rate is found by liner interpolation between two zero coupon T-bonds (cuisp 912833BY and BZ) that closely straddle the maturity of the corporate bond. Knowing the full price, semi-annual coupon rate, number of coupons, default risk free discount rate and exact time to maturity and assuming a recovery fraction of 40%, only one unknown, the probability of survival ρ, remains in equation 1. Iteratively equation 1 is solved for the value of ρ that obtains the full price. This value, along with the above information is all that is needed to solve for the survival interest rate sensitivity measure (equation 2), survival convexity (equation 4), pure interest rate sensitivity measure (equation 3) and the pure interest convexity.

All individual sensitivity measures are then value weighted to obtain the corresponding portfolio measures. This information is used to measure the nine hedge ratios for each of the portfolios in the sample. For modified Macaulay and modified Macaulay with convexity hedge ratios (hedge ratios 1 and 2), $K = \Delta R/\Delta i$ was assumed to equal one, where R is the yield on corporate bonds. This is the assumption that is implicitly made when using modified Macaulay duration to hedge bonds subject to default risk. This implies that yield spreads are constant with respect to a change in the underlying level of interest.

For the survival, and survival with convexity (hedge ratios 5 and 6) $K = \Delta\rho/\Delta i$ is measured by again assuming constant yield spreads. In particular, for a 100 basis point increase in the corporate yield R, the price response that results from the usual bond pricing model is noted. Then that change in the survival probability ρ is found that leads to the same price response in Eq (1). This is done for each bond and the value weighted average of these $\Delta\rho$s is then used to measure the K in hedge ratios 5, and 6 where $\Delta i = .01$.

For the survival and pure interest rate interaction models (hedge ratios 7, 8 and 9), $\Delta\rho$ in K is again measured as the change in ρ necessary to obtain the same price change for equation 1 as in the traditional bond pricing equation in response to a

100 basis point change in corporate bond yield, only now $\Delta\rho$ is calculated after applying a 100 basis point change in the pure interest rate. Again this is done for each bond in the portfolio and the weighted average is used in ratios 7, 8 and 9. In this way all measures of interest rate sensitivity have exactly the same assumption regarding the behavior of yield spreads in response to a change in the general level of interest so the performance of each alternate hedge ratio is comparable. Finally, the variance of interest rates appears in all interaction models. This value is empirically measured using the prior two years of monthly one year t-bill interest rates from the *Citibank Econometric Data Base.*[15]

Table 4 reports summary statistics for the high (AA) and low (B) credit quality bond portfolios along with the corresponding hedging instrument. Reflecting bond market conditions of the time, "AA" bonds have a longer maturity but lower underlying zero coupon yield than "B" rated bonds. Notice that the pure interest and pure interest convexity measures are smaller than the corresponding Macaulay values. This is consistent with Leland (1994) who finds that effective duration and convexity are less than Macaulay duration and convexity. Consistent with our earlier numerical example of Figures 1 and 2, the high grade bond portfolio has larger pure interest and survival sensitivities and larger pure interest and survival convexity than the low grade bond portfolio.

Tables 5 and 6 report the performance of the nine hedge ratios for the high and low credit bond portfolios respectively. The first panel in Tables 5 and 6 reports the full portfolio results while the second and third panel report results for two sub-portfolios, the first formed by choosing only odd numbered and the second by choosing only even numbered bonds. These nine bond portfolios indicate the importance of diversification in reducing unique credit risk.

[Table 4 about here]

First we discuss the full portfolio results of Tables 5 and 6. Column two of each table reports the nine hedge ratios. Classified by size, there appears to be four classes of ratios, modified Macaulay, with and without convexity, pure interest, with and without convexity, survival, with and without convexity, and the three interaction ratios, pure interest and survival, pure interest with convexity and survival, and pure interest and survival both with convexity. Ranked by size, the smallest ratios are the pure interest ratios, followed by the modified Macaulay, interaction and finally survival ratios. Altogether a considerably wide range of values are obtained. Notice that virtually all hedge ratios are less than one, indicating that credit riskless bonds are more interest rate sensitive than bonds subject to credit risk. The two exceptions are the survival ratios, their values are always greater than one. These observations suggest that improvements in hedging strategies may be available, depending on whether credit risky bonds have a consistently greater (less) response to a change in the level of interest than that suggested by the duration based hedge ratio now commonly used.

[Table 5 and 6 about here]

Columns three and four show the gain (loss) from the hedge and cash positions respectively. For this time period, interest rates have increased, resulting in gains on the hedging instrument and losses on the cash position. Gains on the hedging instrument correspond with the hedge ratio, the larger the hedge ratio, the larger the gain. The appropriateness of these gains can be judged with reference to the size of losses on the cash position. Ideally, the cash losses should be the same as the hedge gains. From this point of view, hedging gains are "too large" for high credit bonds (Table 5) and "too small" for low credit bonds (Table 6). Cash losses are much greater in Table 6, showing that at least in this case bonds subject to greater credit risk have a greater price response to a change in the level of interest. The summary portfolio information of Table 4 reveals that this result cannot be attributed to either a difference in maturity or a twist in the underlying yield curve.

The low quality portfolio has a shorter maturity, only 9.5 years, compared to a maturity of 17 years for the high quality portfolio. Other things equal, we would expect the low quality portfolio to respond less to an increase in interest rates. In addition, the change in the underlying yield curve as revealed by the change in yield of the credit riskfree hedging instrument shows that at 9.5 years, riskfree yields increased 48 basis points, while at 17 years maturity, riskfree yields increased 71 basis points. Once again we would expect a greater price response in the high quality bond portfolio which is opposite of what has happened.

This suggests that indeed, low credit bonds do respond more to a change in the underlying yield curve than high credit quality bonds and may well prove to be the reason why Macaulay duration based hedge ratios perform well for high quality bonds, but poorly for low credit quality bonds. If this is the case, then hedge ratios that are greater than the modified Macaulay duration based hedge ratio would increase hedging effectiveness for low credit quality bonds.

Column six reports the net gain (loss) from the hedge strategy. All hedge ratios were "too large" for the high credit quality bond portfolio and "too small" for the low credit quality bond portfolio as there were overall gains in Table 5 but overall losses in Table 6. As a result the "best" hedge ratio for the high credit bond portfolio was the smallest ratio, pure interest; and the best ratio for the low credit quality bond portfolio was the largest hedge ratio, the survival with convexity hedge ratio. Certainly it is too early to draw any generalizations from this single result. Nevertheless, it is interesting to note that this new measure of interest rate sensitivity, the survival based sensitivity measure, holds potential for improvements in hedging effectiveness for low credit quality bond portfolios.

Finally, column seven yields information about the degree of protection provided by each hedge strategy. Ideally this value should be 1, representing 100 percent protection where the net difference column would be zero. Comparing the performance of the modified Macaulay based duration hedge ratio with the most favorable performing hedge ratio will reveal the potential for improving hedging effectiveness. The degree of protection improves by about ten percent for the high credit quality bond portfolio, and by about twelve percent for the low credit quality bond portfolio. For a million dollar investment, this would represent a \$4,000 and a \$10,500 improvement in the hedge performance for the high and low credit bond portfolio respectively[16]. This facts highlight the potential for improvement in hedging effectiveness.

Panels two and three of Tables 5 and 6 provide the same information for sub-portfolios. The interesting fact revealed here is the importance of diversification in corporate bond investments. For both high and low credit quality bond portfolios, cash losses are larger and smaller in the two sub-portfolios than for the full portfolio results, but the differences are much larger in the case of the low credit quality portfolio. For the high credit quality bond portfolio, the performance of each hedging strategy is comparable to the full portfolio results of panel one, but for the low credit quality bond portfolio, performance is greatly affected by a smaller portfolio size. These results suggest that diversification plays an important role in reducing the unique aspects of credit risk, particularly for the low credit bond investments.

IV Summary and Suggestions for Further Research

A new interest rate sensitivity measure, the survival measure is derived. Along with pure interest and modified Macaulay interest rate sensitivity measures we derive nine different hedge ratios, seven of which are new. These hedge ratios exhibit considerable differences in values when applied to a portfolio of high credit (AA) and low credit (B) bonds indicating a potential to improve hedging effectiveness through use of hedge ratios other than the commonly used Macaulay duration based hedge ratios. In particular, prior empirical work has noted the inadequacy of the Macaulay based hedge ratios in hedging lower credit quality bond portfolios. The survival based hedge ratios seem to offer significant improvements in modeling the price response of low credit quality bond portfolios to interest rate changes. In turn this can lead to improvements in hedging performance. Finally, this work indicates that diversification plays an important role in reducing unique credit risk, particularly for low credit quality bonds.

The next step is to replicate the empirical example over different time periods. Ten new cash portfolios will be constructed from May 31, 1990 to November 30, 1994. This time period represents the last credit and interest rate cycle, where both interest rate and with a lag, bankruptcies rose and then fell. For each time period two diversified 18 bond cash portfolios will be formed as was done here, one for a portfolio of "AA" rated bonds and another for a portfolio of "B" rated bonds. Sub-portfolio as well as the full portfolio hedging performance will be evaluated to generate information about the usefulness of diversification of credit risk. Therefore each of the nine hedge ratios will be evaluated a total of 33 times, including 11 full portfolio evaluations, on a total of 396 bonds. Overall performance of each hedge ratio should allow for drawing general conclusions.

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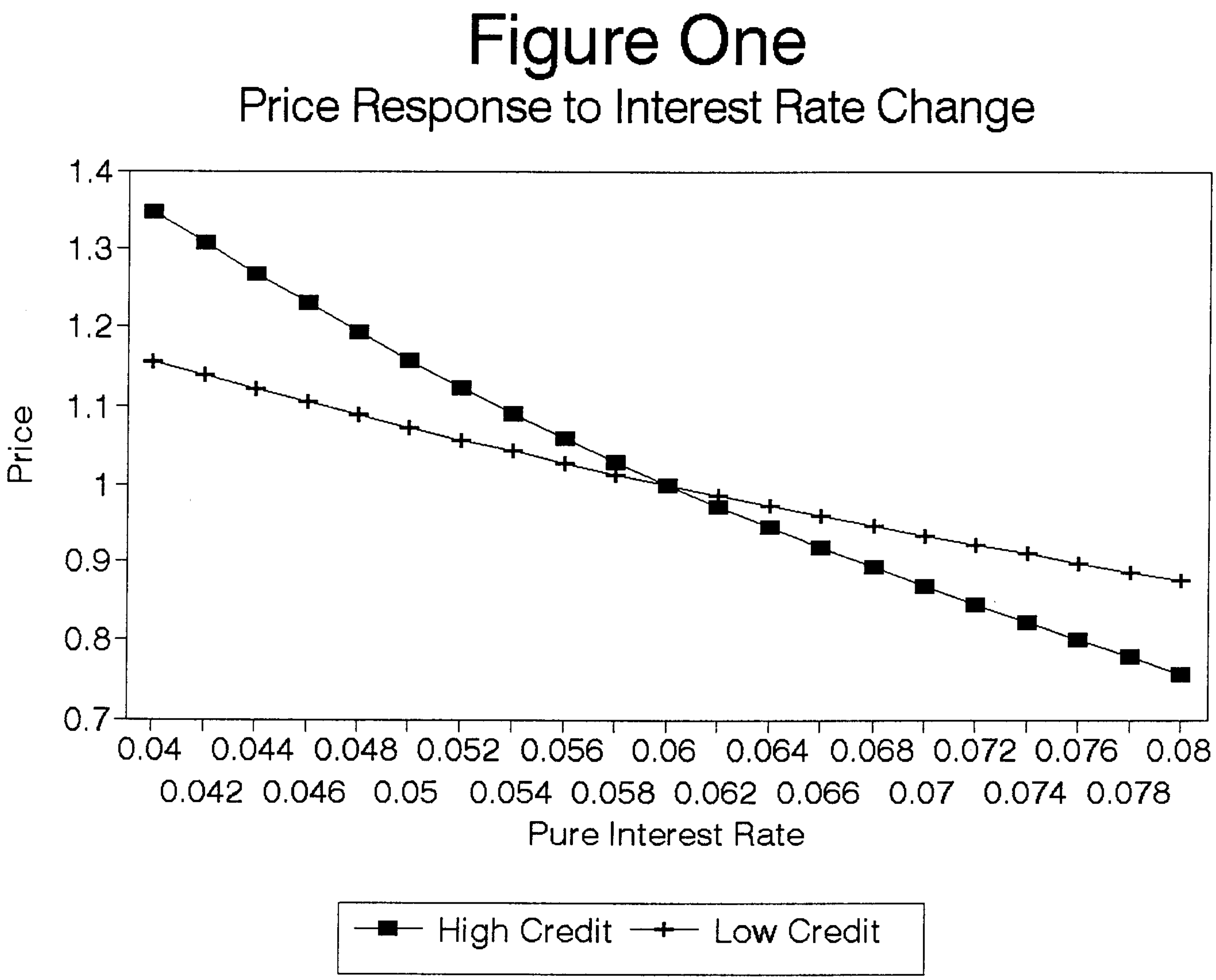
Figure One
Price Response to Interest Rate Change
Price
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.04
0.042
0.044
0.046
0.048
0.05
0.052
0.054
0.056
0.058
0.06
0.062
0.064
0.066
0.068
0.07
0.072
0.074
0.076
0.078
0.08
Pure Interest Rate
High Credit
Low Credit

Figure Two

Price Response to Survival Rate Change

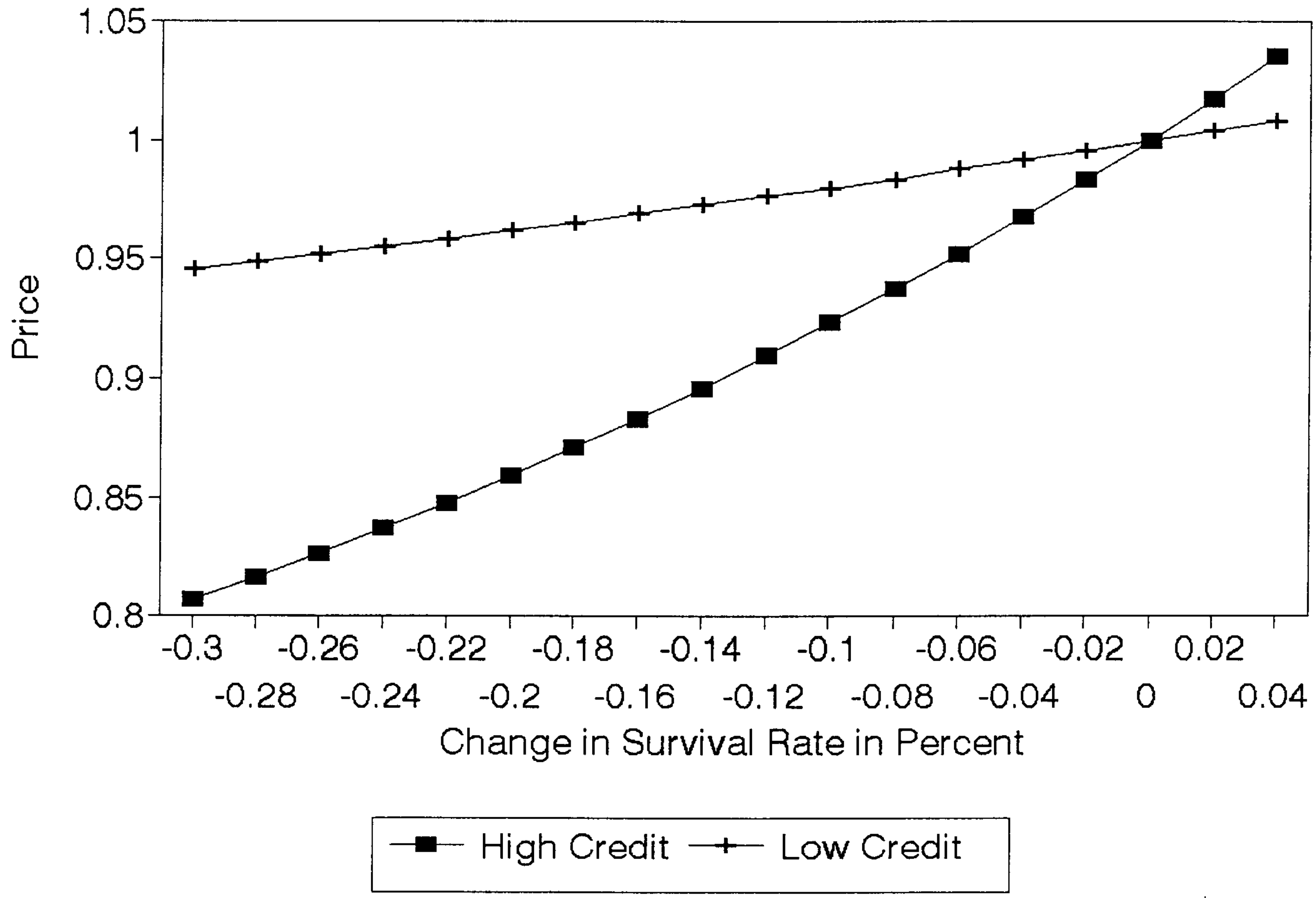

Table 1
Pricing Errors in Response to Yield, Underlying Interest
or Survival Parameter Change for High Credit Quality Bonds

Change, (Semi-annual, %)	Mac. (Per $)	Mac. Conv. (Per $)	Pure Interest (Per $)	Pure Interest Conv. (Per $)	Survival (Per $)	Survival Conv. (Per $)
-.0256	.10447	.01735	.10280	.01704	1.6696	-3.54535
-.0128	.02379	.00201	.02341	.00197	.65626	-.64748
-.0064	.00569	.00024	.00560	.00024	.22266	-.10328
-.0032	.00139	.00003	.00137	.00003	.06663	-.14850
-.0016	.00034	.00000	.00034	.00001	.01837	-.00201
-.0008	.00008	.00000	.00009	.00000	.00483	-.00027
-.004	.00003	.00000	.00002	.00000	.00124	-.00003
.0002	.00000	.00000	.00000	.00000	.00031	-.00001
0	.00000	.00000	.00000	.00000	.00000	.00000
.0002	.00001	.00000	.00001	.00000	.00032	.00000
.0004	.00002	.00000	.00002	.00000	.00131	.00003
.0008	.00009	.00000	.00008	.00000	N/A	N/A
.0016	.00034	.00000	.00033	.00001	N/A	N/A
.0032	.00134	-.00002	.00132	-.00002	N/A	N/A
.0064	.00522	-.00022	.00514	-.00022	N/A	N/A
.0128	.02003	-.00175	.01972	-.00173	N/A	N/A
.0256	.07398	-.01315	.07284	-.01292	N/A	N/A

Table 2
Pricing Errors in Response to Yield, Underlying Interest
or Survival Parameter Change for Low Credit Quality Bonds

Change, (Semi-annual, %)	Mac. (Per $)	Maca. Conv. (Per $)	Pure Interest (Per $)	Pure Interest Conv. (Per $)	Survival (Per $)	Survival Conv. (Per $)
-.0256	.04976	.00711	.02892	.00348	.28312	-.36465
-.0128	.01149	.00083	.00677	.00041	.09879	-.06315
-.0064	.00276	.00010	.00163	.00004	.03077	-.00972
-.0032	.00068	.00001	.00040	.00000	.00875	-.00137
-.0016	.00017	.00000	.00010	.00000	.00235	-.00018
-.0008	.00005	.00000	.00003	.00000	.00060	-.00003
-.004	.00001	.00000	.00001	.00000	.00015	-.00001
.0002	.00000	.00000	.00000	.00000	.00004	.00000
0	.00000	.00000	.00000	.00000	.00000	.00000
.0002	.00001	.00000	.00000	.00000	.00003	.00000
.0004	.00001	.00000	.00000	.00000	.00016	.00000
.0008	.00004	.00000	.00002	.00000	N/A	N/A
.0016	.00017	.00000	.00010	.00000	N/A	N/A
.0032	.00066	-.00001	.00039	-.00001	N/A	N/A
.0064	.00257	-.00010	.00154	-.00005	N/A	N/A
.0128	.00993	-.00074	.00599	-.00037	N/A	N/A
.0256	.03709	-.00556	.02263	-.00281	N/A	N/A

Table 3
Alternate Hedge Ratios*

Type	Hedge Ratio
1) modified Macaulay	$-D_cB_cK/D_hB_h$
2) modified Macaulay with Convexity	$-[D_cB_cD_hB_hK+2(D'_cB_cD'_hB_hK^2)Var(\Delta i)]/[(D_hB_h)^2 + 2(D'_hB_h)^2Var(\Delta i)]$
3) Pure Interest	$-d_cB_c/D_hB_h$
4) Pure Interest with Convexity	$-[d_cB_cD_hB_h+2(d'_cB_cD'_hB_h)Var(\Delta i)]/[(D_hB_h)^2 + 2(D'_hB_h)^2Var(\Delta i)]$
5) Survival	$-S_cB_cK/D_hB_h$
6) Survival with Convexity	$-[S_cB_cD_hB_hK+2(S'_cB_cD'_hB_hK^2)Var(\Delta i) /[(D_hB_h)^2 + 2(D'_hB_h)^2Var(\Delta i)]$
7) Pure Interest and Survival	$-(d_cB_c +S_cB_cK) /D_hB_h$
8) Pure Interest with convexity and Survival	$-[d_cB_cD_hB_h+S_cB_cD_hB_hK+2(d'_cB_cD'_hB_h) Var(\Delta i)/[(D_hB_h)^2 + 2(D'_hB_h)^2Var(\Delta i)]$
9) Pure Interest and Survival, both with Convexity (Kitchen Sink)	$-[d_cB_cD_hB_h+S_cB_cD_hB_hK+2(d'_cB_cD'_hB_h + S'_cB_cD'_hB_hK^2)Var(\Delta i)] / [(D_hB_h)^2 + 2(D'_hB_h)^2Var(\Delta i)]$

* Note that S is the survival measure, d is the pure interest measure, D is modified Macaulay duration and B is the market price of a bond. Subscripts c and h refer to the cash and hedging instruments. Finally, symbols with a prime mark refer to the second derivative (convexity) of the respective measure.

Table 4
Summary Information of Portfolio Formed
November 30, 1989

Statistic	High Quality Portfolio	Hedging Instrument	Low Quality Portfolio	Hedging Instrument
No. Coupons (#)	34	34	19	19
Coupon Rate (%)	8.687	9.375	12.969	8.50
Yield (%)	9.016	7.977	14.837	8.220
Full Price (Per $)	1.00299	1.152889	.951	1.021652
Zero Coupon Yield (%)	7.97	N/A	8.058	N/A
Survival Rate (%)	.999132	N/A	.990985	N/A
Macaulay Duration (Semi-annual)	15.78603	17.716	10.6047	13.38
Convexity (Semi-Annual)	196.8203	214.8	70.40191	108
Pure Interest Sensitivity (Semi-Annual)	14.63556	N/A	8.79645	N/A
Convexity (Semi-Annual)	166.9791	N/A	54.90686	N/A
Survival Sensitivity (Semi-Annual)	103.5002	N/A	32.73725	N/A
Convexity (Semi-Annual)	314.1618	N/A	22.57864	N/A
$\Delta\rho$ for a 1% ΔR	.002135	N/A	.004615	N/A
and Δ I	.000136	N/A	.000824	N/A

Table 5
Hedging Results-High Quality Portfolio
Formed November 30, 1989 and Tested May 31, 1990

Panel 1 (Full)

(1) Hedge Type	(2) Hedge Ratio	(3) Change in Hedge Instrument (Per $)	(4) Change in Cash Portfolio (Per $)	(5) Difference (Per $) (3-4)	(6) Protection (In %) (1-5/4)
modified					
Macaulay.	.8795	.0597	-.0384	.0213	1.5531
Convexity.	.8992	.0610	-.0384	.0226	1.5879
Pure					
Interest.	.8195	.0556	-.0384	.0172	1.4471
Convexity.	.8376	.0569	-.0384	.0184	1.4791
Survival.	1.3153	.0893	-.0384	.0508	2.3226
Convexity.	1.3347	.0906	-.0384	.0522	2.3569
Pure Interest and Survival.	.9216	.0626	-.0384	.0241	1.6274
Pure Interest, Convexity and Survival.	.9211	.0625	-.0384	.0241	1.6265
Pure Interest, Survival, both with Convexity.	.9212	.0625	-.0384	.0241	1.6267

Panel 2 (Odd)

(1) Hedge Type	(2) Hedge Ratio	(3) Change in Hedge Instrument (Per $)	(4) Change in Cash Portfolio (Per $)	(5) Difference (Per $) (3-4)	(6) Protection (In %) (1-5/4)
modified					
Macaulay.	.9523	.0646	-.0428	.0218	1.5092
Convexity.	.9739	.0661	-.0428	.0233	1.5434
Pure					
Interest.	.8846	.0601	-.0428	.0172	1.4019
Convexity.	.9043	.0614	-.0428	.0186	1.4331
Survival.	1.3938	.0946	-.0428	.0518	2.2089
Convexity.	1.4140	.0960	-.0428	.0532	2.2409

Pure Interest and Survival	.9731	.0661	-.0428	.0232	1.5422
Pure Interest, Convexity and Survival.	.9729	.0660	-.0428	.0232	1.5418
Pure Interest, Survival, both with Convexity.	.9730	.0661	-.0428	.0232	1.5420

Panel 3 (Even)

modified Macaulay.	.8068	.0548	-.0340	.0207	1.6093
Convexity.	.8246	.0560	-.0340	.0219	1.6448
Pure Interest.	.7545	.0512	-.0340	.0172	1.5050
Convexity.	.7709	.0523	-.0340	.0183	1.5377
Survival.	1.1989	.0814	-.0340	.0474	2.3915
Convexity.	1.2158	.0825	-.0340	.0485	2.4252
Pure Interest and Survival.	.8620	.0585	-.0340	.0245	1.7194
Pure Interest, Convexity and Survival.	.8613	.0585	-.0340	.0244	1.7180
Pure Interest, Survival, both with Convexity.	.8614	.0585	-.0340	.0244	1.7182

Table 6
Hedging Results-Low Quality Portfolio
Formed November 30, 1989 and Tested May 31, 1990

Panel 1

(1) Hedge Type	(2) Hedge Ratio	(3) Change in Hedge Instrument (Per $)	(4) Change in Cash Portfolio (Per $)	(5) Difference (Per $) (3-4)	(6) Protection (In %) (1-5/4)
modified					
Macaulay.	.7463	.0225	-.0855	-.0623	.2632
Convexity.	.7149	.0216	-.0855	-.0639	.2521
Pure					
Interest.	.6393	.0193	-.0855	-.0662	.2254
Convexity.	.6122	.0185	-.0855	-.0670	.2159
Survival.	1.0943	.0330	-.0855	-.0525	.3859
Convexity.	1.0991	.0331	-.0855	-.0524	.3876
Pure					
Interest and Survival.	.8078	.0244	-.0855	-.0611	.2848
Pure					
Interest, Convexity and Survival.	.8073	.0243	-.0855	-.0612	.2847
Pure					
Interest, Survival, both with Convexity.	.8075	.0243	-.0855	-.0612	.2847

Panel 2

modified					
Macaulay.	.7928	.0239	-.0235	.0004	1.0153
Convexity.	.7593	.0229	-.0235	-.0007	.9724
Pure					
Interest.	.6870	.0207	-.0235	-.0028	.8798
Convexity.	.6580	.0198	-.0235	-.0037	.8427
Survival.	.9372	.0283	-.0235	.0047	1.2002
Convexity.	.9402	.0283	-.0235	.0048	1.2041

Pure Interest and Survival.	.7883	.0238	-.0235	.0002	1.0095
Pure Interest, Convexity and Survival.	.7878	.0237	-.0235	.0002	1.0089
Pure Interest, Survival, both with Convexity.	.7879	.0238	-.0235	.0002	1.0090

Panel 3

modified Macaulay.	.6977	.0210	-.1528	-.1318	.1376
Convexity.	.6682	.0201	-.1528	-.1327	.1318
Pure Interest.	.5893	.0178	-.1528	-.1351	.1163
Convexity.	.5644	.0170	-.1528	-.1358	.1113
Survival.	1.2054	.0363	-.1528	-.1165	.2378
Convexity.	1.2122	.0365	-.1528	-.1163	.2391
Pure Interest and Survival.	.8117	.0245	-.1528	-.1283	.1601
Pure Interest, Convexity and Survival.	.8111	.0245	-.1528	-.1284	.1600
Pure Interest, Survival, both with Convexity.	.8115	.0245	-.1528	-.1284	.1600

Appendix 1

The objective is to minimize the price variance of the beginning value of a hedge portfolio. All notation used here is the same as used in the text. The hedge portfolio is

$$V_h = B_c + NB_h$$

Since $\Delta\rho = -K\Delta i + \varepsilon$, $\Delta B_c \cong S_c B_c \Delta\rho + S'_c B_c (\Delta\rho)^2, \Delta B_h \cong -D_h B_h \Delta i + D'_h B_h (\Delta i)^2$ then

$$\Delta V_h \cong S_c B_c(-K\Delta i + \varepsilon) + S'_c B_c(-K\Delta i + \varepsilon)^2 + N[-D_h B_h \Delta i + D'_h B_h(\Delta I)^2$$

Where $K = \Delta\rho/\Delta i$, S_c is equation 2 (adjusted by dividing by B_c), the survival "duration", S'_c is equation 5, the survival convexity (adjusted by dividing by 2) , D_h is Macaulay modified duration and D'_h is Macaulay convexity. Since $E(\varepsilon) = 0$, $E(\Delta i) = 0$ and $E(\Delta i,\varepsilon) = 0$, and using the properties of the expectations operator (E), the variance of changes in the value of the hedge portfolio is,

$$\mathrm{Var}[\Delta V_h] = \mathrm{Var}(\Delta i)\,[(S_c B_c\text{-}K)^2 + 2S_c B_c K N D_h B_h + (ND_h B_h)^2] + \mathrm{Var}(\Delta i)^2\,[(S'_c B_c K^2)^2 + 2S'_c B_c K^2 N D'_h B_h + (ND'_h B_h)^2] + 2\mathrm{Cov}(\Delta i,\Delta i^2)[(S_c B_c\text{-}K + N\text{-}D_h B_h)\,(S'_c B_c K^2 + ND'_h B_h)] + (S_c B_c)^2\,\mathrm{Var}(\varepsilon) + (S'_c B_c)^2\,\mathrm{Var}(\varepsilon)^2 .$$

We assume that Δi is I.I.D. normal so Δi^2 is distributed Chi-Square. According to standard distribution theory, if $\mathrm{Var}(\Delta i) = \sigma^2$ then $\mathrm{Var}(\Delta i^2) = 2\sigma^4$[17] Finally, note that $\mathrm{Var}(\varepsilon)^2$ is assumed $=0$ and $\mathrm{Cov}(\Delta i, \Delta i^2) = 0$. Taking the first derivative with respect to N, setting the result equal to zero and solving for N we obtain,

$$N = -[S_c B_c D_h B_h K + 2(S'_c B_c D'_h B_h K^2)\,\sigma^2{}_{\Delta i}]/[(D_h B_h)^2 + 2(D'_h B_h)^2 \sigma^2{}_{\Delta i}]$$

This is the survival with convexity hedge ratio as reported in Table 3.

---

[1] This classification scheme is based on Van Horne (1994), with a refined term, credit risk, used as a substitute for default risk. Default is the ultimate consequence of credit risk, but credit risk is a much broader term. It includes price variations due to changes in the likelihood of default, caused by either idiosyncratic or market wide factors and changes in valuation of recoveries in the event of default. By this definition, even bonds that have defaulted are still subject to credit risk as the valuation of recoveries are still uncertain.

[2] For an explanation of the duration based hedge ratio, see Chaing and Kolb (1981). For a review of the use of immunization in bond portfolio management, consult Bierwag, Kaufman, and Toevs (1983).

[3] A flat term structure is assumed in this analysis since a comparison between new and existing hedge ratios based on Macaulay duration is desirable. Later work can extend measures derived here for non-flat term structures.

[4] The one period probability of survival is carried to the exponent of time as this is theoretically consistent with normal bond pricing. For details of the implications of this time structure of survival, see Rubicheck and Myers (1966).

[5] The partial derivative of the bond's price with respect to the recovery fraction was not considered. One would expect that as the pure rate of interest changes, the brunt of its effect on credit risk of bonds that have not yet defaulted would be on a reassessment of the probability of survival rather than recoveries in the event of default. Leland (1994) does consider the effect of recovery rates in the case of defaulted bonds, showing that interest rate sensitivity may turn negative in that case.

[6] To clearly see what is involved, consider the following. What we want is the total derivative,

$$dB/dI = \partial B/\partial I + \partial B/\partial \rho \; d\rho/dI$$

The partial derivatives are contained in equations 2 and 3, but how to model $d\rho/dI$? We need an equation that is independent of the Jonkhart (1979) model. One possibility (Bierman and Hass (1975) for instance) is to use an expression for the credit spread as the credit spread is determined by both ρ and I. But of course the credit spread also includes another parameter, the default risky yield, r. So we end up with

$$d\rho/dI = \partial \rho/\partial I + \partial \rho/\partial r \; d\rho/dr$$

Now how do we model $d\rho/dr$? Again we need another equation independent of the other two. Well, option pricing theory says that the probability of default $(1-\rho)$ is related to the value of the firm, which is related to both the bondholder yield, r. But of course the shareholder return , R is introduced as well. We end up with

$$d\rho/dr = \partial \rho/\partial r + \partial \rho/\partial R \; d\rho/dR$$

Now we need $d\rho/dR$, and so on.

[7] In a variation of the modified Macaulay duration hedge ratio, K is measured as an OLS regression coefficient where past corporate bond yields are regressed on T-Bond yields. This K would represent the past responsiveness of the corporate bond yield to past changes in T-Bond yields. Toevs and Jacob (1987) test the performance of this adjustment and conclude that unadjusted duration hedges dominate the more elaborate regression adjusted duration hedge ratios.

[8] Altman and Kishore (1996) find that senior bonds and investment grade bonds have higher recovery rates than average, and subordinate and below investment grade bonds have recovery rates close to the average rate. However, subdividing the sample of defaulted bonds greatly reduces the sample size since default experience seems concentrated in the original issue below investment grade segment of the bond market. In any event, a replication of the following analysis using higher recovery rates for the high grade bond obtains the same results reported here.

[9] It may appear to the reader that there is a tenth model, a survival with convexity and pure interest rate hedge ratio. However, deriving this model from basic principles as outlined in the text will reveal that the resulting hedge ratio is the same as the survival and pure interest hedge ratio.

[10] Details of any other derivation is available from the author upon request.

[11] Minimizing the price variance of the terminal value of the hedge portfolio is another viable objective, but performance evaluation will be more difficult since performance will be in part dependent upon an interest rate forecast as well as the appropriateness of the interest rate sensitivity measure. As a result, only beginning value variance minimization is considered.

[12] Alternatively we could use fixed income futures contracts as traded on the CBOT. However, performance of the hedge will depend in part on which bond is actually delivered. Investors who are short in the fixed income futures contracts have a choice among a schedule of deliverable bonds, so obviously they will chose to deliver the "cheapest" bond. Unfortunately, as interest rates are stochastic, the "cheapest to deliver" bond will change from time to time during the life of the hedge. Performance evaluation will be more difficult since performance will be in part dependent upon an interest rate forecast (made to

forecast the "cheapest to deliver bond") as well as the appropriateness of the interest rate sensitivity measure. As a result, the T-Bond (or T-Note) that has the closest maturity to the average maturity of the corporate bond portfolio is used as the hedging instrument.

[13] Very few single B (2) and double B (6) rated bonds were non-callable. As we are interested in developing hedge ratios for below investment grade bonds, we are forced to examine call protected issues. Nevertheless the empirical results were also replicated for non-callable bonds, only now the "low credit" bond portfolio consisted of triple B bonds. These results were qualitatively the same as the call protected portfolios reported in the text.

[14]Note that the semiannually convexity measure is already divided by 2 since it will be used as the second term of the Taylor order expansion to construct hedge ratios.

[15] Different values of the variance of interest rates were tested. It was found that the hedge ratio values were not sensitive to this information, so the Citibank information was used as it was readily available.

[16]For debt portfolios, a million dollar portfolio is a *small* portfolio.

[17]Since $\Delta i \sim N(0, \sigma^2)$ then $\Delta i/\sigma \sim N(0,1)$. $Var(\Delta i/\sigma) = 1$. Then $\sigma^2\, Var(\Delta i/\sigma) = \sigma^2$ so $Var(\Delta i) = \sigma^2$.
Similarly
$\Delta i^2 \sim \chi^2$ and $\Delta i^2/\sigma^2 \sim \chi^2_1$ and $Var(\Delta i^2/\sigma^2) = 2$. Then $\sigma^4\, Var(\Delta i^2/\sigma^2) = 2\sigma^4$ so $Var(\Delta i^2) = 2\sigma^4$.