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Abstract

We consider the demand for state contingent claims in the presence of a zero-mean, non-hedgeable background risk. An agent is defined to be generalized risk averse if he/she reacts to an increase in background risk by choosing a demand function for contingent claims with a less steep slope. We show that the conditions for standard risk aversion: positive, declining absolute risk aversion and prudence are necessary and sufficient for generalized risk aversion.

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D52, D81, G11."

1 Introduction

Recent advances in the theory of risk bearing have concentrated on the effect of a non-hedgeable background risk on the risk aversion of an agent to a second independent risk. For example, Gollier and Pratt (1996) define a rather general class of utility functions such that risk-averse individuals become more risk averse towards a risk, when a second, independent unfair background risk is added. They compare the risk aversion of an agent with no background risk to that of an agent who faces the background risk. They term the set of functions under which the agent becomes more risk averse, the class of "risk vulnerable" utility functions. The set of risk vulnerable functions is larger than the set of proper risk averse functions introduced earlier by Pratt and Zeckhauser (1987); they considered utility functions such that the expected utility of an undesirable risk is decreased by the presence of an independent, undesirable risk. Kimball (1993) has considered the effect of the [even larger set] of expected marginal utility increasing background risks. This led him to define the more restrictive class of standard risk averse utility functions. Standard risk aversion characterises those functions where the individual responds to an expected marginal utility increasing background risk by reducing the demand for a marketed risk. Kimball shows that standard risk averse functions are characterized by positive, decreasing absolute risk aversion and absolute prudence. The set of standard risk averse functions is a subset of the set of proper risk averse functions which in turn are a subset of the risk vulnerable functions, as discussed by Gollier and Pratt (1996, pp 1118-9). In a related paper, Eeckhoudt, Gollier and Schlesinger (1996) extend this analysis by considering a rather general set of changes in background risk, which take the form of first or second order stochastic dominance changes. They establish a set of very restrictive conditions on the utility function such that agents become more risk averse when background risk increases in this sense.

The purpose of this paper is twofold. First we consider a smaller set of increases in background risk than Eeckhoudt, Gollier and Schlesinger (1996) and derive less restrictive conditions for an increase in background risk to increase the derived risk aversion of agents. We restrict the set of increases in the risk of background income y, with $E(y) = 0$, to simple increases (see also Eeckhoudt, Gollier and Schlesinger (1995)). A simple increase in background risk is a change Δy to y such that $E(\Delta y) = 0$ and $\Delta y \leq [=][\geq] 0$ for $y < [=] > y_0$ for some y_0. We derive a necessary and sufficient condition on the utility function, for a simple increase in background risk to make the agent more risk averse. We show that standard risk aversion is sufficient, but not necessary for a simple increase in background risk to increase derived risk aversion.

The second and the main purpose of the paper is to investigate restrictions on utility functions which guarantee a more risk averse behaviour in the presence of an increased, independent background risk, when the agent faces a choice between state contingent claims. Changes in risk averse behaviour are reflected in the slope of the demand function for contingent claims. We assume, quite generally, that the agent can buy claims on consumption in various states of nature. The agent observes a set of prices for state contingent claims. The set of probability deflated prices is denoted $[\phi]$. The higher is ϕ for a given state, the lower is the agent's demand for claims contingent on that state. In other words, the demand function for state contingent claims has a negative slope. A natural extension of the notion of more risk averse behaviour to the case of contingent claims demand is as follows. An agent is more risk averse, if the slope of his/her demand function for contingent claims becomes less steep, at all levels of ϕ.[1] If an agent responds to an increase in background risk by choosing a less steeply sloping demand function at all levels, we call the agent generalized risk averse. The agent's behaviour exhibits generalized risk aversion. In this paper we derive conditions on the utility function for an agent to be 'generalized risk averse'.

We consider the effect of an independent background risk on the demand for contingent claims, using an extension of the analysis of Back and Dybvig (1993), who establish conditions for the optimality of an agent's demand for state contingent claims. We investigate the set of [restrictions on] utility functions such that the agent responds to *monotonic* increases in zero-mean background risk by choosing a demand function that has a less steep slope at all price levels. In the context of this choice problem, we need to further restrict the set of changes in background risk that are considered, to the set of monotonic increases. A monotonic increase in background risk, y is defined as a change Δy, where $E(\Delta y) = 0$ and where $\partial \Delta y / \partial y \geq 0, \forall y$. Hence a monotonic increase in background risk, Δy, is an increase that itself increases with y. The simplest example of a monotonic increase is a proportionate increase where Δy is proportionate to y. Assuming monotonic increases in background risk we find that the set of generalized risk averse

[1] Pratt(1964) has shown that an agent facing the choice between a riskless and a single risky asset buys less of the risky asset when his/her risk aversion increases. We extend this notion to the demand for state-contingent claims.

utility functions is the standard risk averse class of Kimball (1993). Hence risk vulnerability is not sufficient for background risk to reduce the steepness of the slope of the demand function for state contingent claims.

The conditions for standard risk aversion: positive, declining absolute risk aversion, and positive, declining absolute prudence, are sufficient for a monotonic increase in background risk to increase derived risk aversion. They are also sufficient for the slope of the demand function for contingent claims to become less steep. What is more surprising is that these conditions are also necessary for generalized risk aversion. Necessity arises from the fact that the slope of the demand function for contingent claims must become less steep at all possible values of ϕ. As Kimball argues, declining absolute risk aversion and declining absolute prudence are natural attributes of the utility function. They are shared, also, by the HARA class of functions with an exponent less than one. The larger set of risk vulnerable utility functions, used by Gollier and Pratt, is not restrictive enough when we consider the effect of background risk on the slope of the demand function. Our result thus adds to the case for the standard risk averse functions to be the natural class of functions to be used when analysing the impact of background risk.

In section 2, we look again at the effect of an increase in background risk on the risk aversion of the derived utility function. Here we are concerned, as were Eeckhoudt, Gollier and Schlesinger (1996) [EGS] with changes in background risk. However, in order to avoid the restrictive conditions on utility found by EGS, we restrict the analysis to simple increases in background risk. We establish a new condition, a generalisation of the result of Gollier and Pratt, for an increase in background risk to increase risk aversion. In section 3, we then introduce the problem of analysing the slope of the demand function for contingent claims. We present our main result: agents choosing state contingent claims become more risk averse in their choice, if and only if they are standard risk averse, i.e. positive and declining absolute risk aversion and prudence is the necessary and sufficient condition for generalized risk aversion.

2 The Effect of an Increase in Background Risk on Derived Risk Aversion

We assume that background risk, y, has a zero mean, and is bounded from below, $y \geq a$. The size of the background risk is represented by an index s. We also assume, in this section, that changes in background risk are restricted to 'simple increases'. A simple increase in background risk, which Eeckhoudt, Gollier and Schlesinger (1995) term 'a simple spread across y_0', is defined as a change in y, Δy, such that $E(\Delta y) = 0$ and

$$\Delta y \leq [=][\geq]0, \text{ if } y < [=][>]y_0$$

The agent's total income, W, is composed of an income, w, from tradable claims, plus the background risk y, i.e. $W = w + y$. We assume that y is distributed independently of w.

The agent's expected utility, conditional on w, is given by the derived utility function, as defined by Kihlstrom et al. (1981) and Nachman (1982):

$$\nu(w) = E_y[u(W)] \tag{1}$$

where E_y indicates an expectation taken over different outcomes of y. Thus, the agent with background risk and a von Neumann-Morgenstern, concave utility function $u(W)$ acts like an individual without background risk and a concave utility function $\nu(w)$.[2] We assume that the utility function, $u(W)$, is concave, four times differentiable, and $u' \in (0, \infty)$, for $W \varepsilon [\underline{W}, \infty)$, i.e. W has a lower bound $\underline{W}$. The coefficient of absolute risk aversion is defined as $r(W) = -u''(W)/u'(W)$ and the coefficient of absolute prudence as $p(W) = -u'''(W)/u''(W)$. The agent is standard risk averse if $r(W)$ and $p(W)$ are both positive and declining. We will also refer to the absolute risk aversion of the derived utility function. This is defined as $\hat{r}(w) = -\nu''(w)/\nu'(w)$.

We first investigate the question of how an agent's derived risk aversion is affected by a simple increase in background risk. Not surprisingly, the condition for an agent's derived risk aversion to increase, when there is a marginal increase in zero-mean background risk, is stronger than the condition of Gollier and Pratt(1996). This is because the 'risk vulnerability' condition of Gollier and Pratt only considers changes in background risk from zero to a finite level, whereas we consider any changes in background risk. However, the condition we derive is weaker than standard risk aversion.

[2]See, for example , Eeckhoudt, Gollier and Schlesinger (1996), p. 684.

Standard risk aversion is a sufficient, but not a necessary condition, for the increase in background risk to increase the agent's derived risk aversion.

The absolute risk aversion of the agent's derived utility function is defined as the negative of the ratio of the second derivative to the first derivative of the derived utility function with respect to w, i.e.

$$\hat{r}(w) = -\frac{\partial \nu''(w)}{\partial \nu'(w)} = -\frac{E_y[u''(W)]}{E_y[u'(W)]} \tag{2}$$

Note that in the absence of background risk , $\hat{r}(w) = r(W)$, the coefficient of absolute risk aversion of the original utility function. In the proposition that follows, we characterize the behavior of $\hat{r}(w)$ in relation to $r(W)$, and explore the properties of derived risk aversion in the presence of increasing zero-mean background risk.

Proposition 1 *(Derived Risk Aversion and Simple Increases in Background Risk)*

If $u'(W) > 0$ and $u''(W) < 0$, then

$$\frac{\partial \hat{r}(w)}{\partial s} > [=][<] 0, \forall (w, s) \Longleftrightarrow$$

$$u'''(W_2) - u'''(W_1) < [=][>] - r(W)[u''(W_2) - u''(W_1)]$$

,

$$\forall (W, W_1, W_2), W_1 \leq W \leq W_2$$

Proof: See Appendix 1.

In order to interpret the necessary and sufficient condition under which an increase in a zero-mean, background risk will raise the risk aversion of the derived utility function, first consider the special case of small risks. In this case, we have

Corollary 1 *In the case of small risks, Proposition 1 becomes*

$$\hat{r}(w) > [=][<] r(W) \quad \text{iff} \quad \frac{\partial \theta}{\partial W} < [=][>] 0, \forall W$$

where $\theta(W) \equiv u'''(W)/u'(W)$.

Proof: Let $W_2 - W_1 \to 0$. In this case, $u'''(W_2) - u'''(W_1) \to u''''(W)$. Similarly $u''(W_2) - u''(W_1) \to u'''(W)$.

Hence, the condition in Proposition 1 yields, in this case, $u''''(W) < [=][>] - r(W)u'''(W)$. This is equivalent to $\partial\theta/\partial W < [=][>]0, \forall W$.[3] $\square$

In Corollary 1, we define an additional characteristic of the utility function $\theta(W) = u'''(W)u'(W)$ as a *combined* prudence/risk aversion measure. This measure is defined by the product of the coefficient of absolute prudence and the coefficient of absolute risk aversion. The corollary says that for a small background risk derived risk aversion exceeds [is equal to] [is smaller than] risk aversion if and only if $\theta(W)$ decreases [stays constant] [increases] with W. Hence, it is significant that *neither* decreasing prudence *nor* decreasing absolute risk aversion is necessary for derived risk aversion to exceed risk aversion. However, the combination of these conditions is sufficient for the result to hold, since the requirement is that the product of the two must be decreasing. The condition is thus weaker than standard risk aversion, which requires that *both* absolute risk aversion and absolute prudence should be positive and decreasing.

We now apply Proposition 1 to show that standard risk aversion is a sufficient, but not a necessary condition, for an increase in background risk to cause an increase in the derived risk aversion [see also Kimball (1993)]. We state this as

Corollary 2 *Standard risk aversion is a sufficient, but not necessary, condition for derived risk aversion to increase with a simple increase in background risk.*

Proof: Standard risk aversion requires both positive, decreasing absolute risk aversion and positive decreasing prudence. Further, $r'(W) < 0 \to p(W) > r(W)$. Also, standard risk aversion requires $u'''(W) > 0$. It follows that the condition in Proposition 1 for an *increase* in the derived risk aversion can be written as[4]

$$\frac{u'''(W_2) - u'''(W_1)}{u''(W_2) - u''(W_1)} < -r(W_1)$$

[3] For a small, unfair background risk, Gollier and Pratt (1996) derive a necessary and sufficient condition based on declining absolute risk aversion

[4] Note that whenever $r'(W)$ has the same sign for all W, the three-state condition in Proposition 1 (i.e. the condition on W, W_1, and W_2) can be replaced by a two-state condition (a condition on W_1 and W_2).

or, alternatively,

$$p(W_1)\quad\left[\frac{1-\frac{u'''(W_2)}{u'''(W_1)}}{1-\frac{u''(W_2)}{u''(W_1)}}\right] > r(W_1)$$

Since $p(W_1) > r(W_1)$, a sufficient condition is that the term in the square bracket exceeds 1. This, in turn, follows from decreasing absolute prudence, $p'(W) < 0$. Hence, standard risk aversion is a sufficient condition.

To establish that standard risk aversion is not necessary, consider a case that is not standard risk averse. Suppose, in particular, that $u'''(W) < 0, u''''(W) < 0$, that is, the utility function exhibits increasing risk aversion and negative prudence.[5] In this case, it follows from Proposition 1 that $\partial\hat{r}/\partial s > 0, \forall(w,s)$. □

In order to obtain more insight into the meaning of the condition in Proposition 1, consider the case where the increase in background risk raises derived risk aversion. Since

$$\hat{r}(w) = E\left[\frac{u'(W)}{E[u'(W)]}r(W)\right],$$

$$\frac{\partial\hat{r}(w)}{\partial s} = E\left[\frac{u'(W)}{E[u'(W)]}r'(W)y'\right] + E\left[r(W)\frac{\partial}{\partial y}\left[\frac{u'(W)}{E[u'(W)]}\right]\right]$$

The first term is positive whenever r is convex. As we showed in the proof of Proposition 1, it suffices to consider a three-point distribution of background risk. For such a distribution, the second term is positive whenever $r' < 0$. Hence a sufficient condition for $\partial\hat{r}(w)/\partial s \geq 0$ is a declining and convex r.[6]But it could also be sufficient that $r' > 0$ if the convexity is sufficiently high. Therefore, there are utility functions with increasing risk aversion which still imply that simple increases in zero-mean background risk raise the derived risk aversion.

[5]As an example, consider the utility function

$$u(W) = \frac{1-\gamma}{\gamma}\left[A + \frac{W}{1-\gamma}\right]^{\gamma}, \text{where } \gamma \in (1,2), W < A(\gamma-1)$$

This utility function exhibits *increasing* risk aversion and *negative* prudence. Still $\theta(W)$ decreases with wealth even in this case and the derived risk aversion increases with background risk.

[6]See also corollary 1 of Gollier and Pratt (1996)

3 The Effect of Changes in Background Risk on the Optimal Demand Function for Contingent Claims

In this section we investigate the effect of background risk on the agent's demand for state-contingent claims. We derive necessary and sufficient conditions for the utility function to exhibit generalised risk aversion.

We assume that the capital market is perfect and that there exists a set of states such that an agent can buy and sell claims paying one unit of consumption contingent on each state. The price of a claim contingent on some state and divided by the probability density of that state is denoted ϕ. We assume that ϕ is positive and continuous. $\forall K > 0$, the agent can buy a claim, which pays one unit of cash if $\phi > K$, and zero otherwise.[7] Also, the agent can buy claims contingent on ϕ.

Let w be the agent's income from state contingent claims at time 1. The agent chooses the demand function $w = w(\phi)$ subject to the constraint that the cost of acquiring this set of claims is equal to his/her initial endowment. The agent's consumption at the end of the single period, W, is equal to the chosen marketed claim, w, plus an independent, zero-mean background risk y, i.e. $W = w + y$. The background risk affects his/her choice of the function w. We also assume that the agent has sufficient endowment so that it is possible for w to be chosen to obtain $W \geq \underline{W}$ in all states. We assume also certain properties of the utility function. First, we assume that marginal utility has the limits:

$$u'(W) \to \infty \text{ if } W \to \underline{W},$$

$$u'(W) \to 0 \text{ if } W \to \infty.$$

Second, we assume that risk aversion goes to zero at high levels of income, i.e.

$$r(W) \to 0 \text{ if } W \to \infty.$$

This is a reasonable restriction and is satisfied, for example, by the HARA class, with an exponent less than 1.

The agent solves the following maximization problem:

$$\max_{w} E[\nu(w)] \tag{3}$$

[7] See Nachman (1988)

$$\text{s.t.} \quad E\left[(w - w^0)\phi\right] = 0$$

In the budget constraint, $w^0 = w^0(\phi)$ is the agent's endowment of claims, and ϕ is the pricing kernel, which is given exogenously. The pricing kernel is a function that gives the forward price of the claim.[8] Hence $E(\phi) = 1$.

The first order condition for a maximum is

$$\nu'(w) = \lambda\phi, \tag{4}$$

where λ is a positive Lagrange multiplier which reflects the tightness of the budget constraint. Equation (4) holds as an equality since, by assumption, $u'(W) \to \infty$ for $W \to \underline{W}$ and $u'(W) \to 0$ for $W \to \infty$. The demand for claims in equation (4) can be shown to be optimal and unique under some further finiteness restrictions. This follows from the results of Back and Dybvig (1993).[9]

Our aim is to find the necessary and sufficient conditions on the utility function, which guarantee that the agent's demand function becomes less steep when background risk increases. First we define

Definition 1 *An agent is generalized risk averse if his/her demand function for state-contingent claims $w(\phi)$ becomes less steep for all ϕ given an increase in background risk.*

Differentiating equation (4) with respect to ϕ, for a given level of background risk, and dividing by $\lambda\phi$, yields the slope of the demand function

$$\frac{\partial w}{\partial \phi} = \frac{-1/\phi}{\hat{r}(w)}, \forall\phi \tag{5}$$

Suppose that background risk increases the derived risk aversion of the agent, $\hat{r}(w)$. It follows from equation (5) that the background risk affects the slope of the demand function. We now consider the effect of changes in the level of background risk, assuming that the pricing function ϕ is given. From equation (5) it appears at first sight that the slope of the demand function becomes less steep whenever the increase in background risk increases the agent's derived risk aversion. This is not true, however, because a change in background risk, affects $\hat{r}(w)$ both directly and through the induced change in w. This is stated in the following proposition.

[8] In a discrete state space setting ϕ is the probability deflated forward price of a state contingent claim paying one unit if and only if the specific state occurs.

[9] We assume that expected utility is finite for any attainable portfolio of claims. Also, $E[w\phi] < \infty$ for any $\lambda > 0$ and each w satisfying (4) is assumed.

Proposition 2 *For the demand function for contingent claims to become less steep with an increase in background risk (generalized risk aversion), it is necessary, but not sufficient for the absolute risk aversion of the derived utility function to increase with background risk. That is*

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0 \Rightarrow \frac{\partial \hat{r}(w)}{\partial s} \geq 0, \tag{6}$$

but

$$\frac{\partial \hat{r}(w)}{\partial s} \geq 0$$

does not imply

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0.$$

Proof: First, totally differentiating equation (5) with respect to s yields, since $1/\phi$ is a constant,

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] = \frac{1/\phi}{[\hat{r}(w)]^2}\frac{d\hat{r}(w)}{ds}, \tag{7}$$

where

$$\frac{d\hat{r}(w)}{ds} = \frac{\partial \hat{r}(w)}{\partial s} + \frac{\partial \hat{r}(w)}{\partial w}\frac{\partial w}{\partial s}. \tag{8}$$

It follows that, since

$$\frac{1/\phi}{[\hat{r}(w)]^2} > 0,$$

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0 \Leftrightarrow \frac{\partial \hat{r}(w)}{\partial s} + \frac{\partial \hat{r}(w)}{\partial w}\frac{\partial w}{\partial s} \geq 0. \tag{9}$$

Given the budget constraint, $\partial w/\partial s$ has to be positive in some states and negative in others. It follows immediately that $\frac{\partial \hat{r}(w)}{\partial s} \geq 0$ is not sufficient to ensure that

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0.$$

Now to establish necessity, suppose that

$$\frac{d}{ds}\left[\frac{\partial w}{\partial \phi}\right] \geq 0$$

for all ϕ, then since

$$\frac{\partial \hat{r}(w)}{\partial w}\frac{\partial w}{\partial s}$$

can be negative for some states, $\frac{\partial \hat{r}(w)}{\partial s} \geq 0$ is a necessary condition.□ Having

shown that increased derived risk aversion is a necessary, but not sufficient condition for generalized risk aversion, we can now establish our main result. In order to analyse the impact of background risk on the slope of the agent's demand function for contingent claims we need to make stronger assumptions. Regarding the background risk we now assume monotonic changes in background risk. This is a somewhat stronger than the previous assumption of simple increases in background risk. First we define monotonic increases in background risk:-

Definition 2 *(Monotonic Increases in Background Risk)*

Let $y_i(s)$ denote a realisation $i = 1, ..., j$ of background risk income, given the level of background risk, s. Suppose that

$$y_1(s) \leq y_2(s) \leq ... \leq y_i(s) \leq ... \leq y_j(s)$$

with $y_i(0) = 0, \forall i$. Then increases in background risk are monotonic if for any $\bar{s} > s \geq 0$,

$$y_1(\bar{s}) - y_1(s) \leq y_2(\bar{s}) - y_2(s) \leq ...y_i(\bar{s}) - y_i(s) \leq ... \leq y_j(\bar{s}) - y_j(s)$$

The effect of assuming monotonic increases in background risk is that the rank order of the outcomes $y_1, y_2, ...$ is preserved under a monotonic increase in background risk.

Proposition 3 *(Generalized Risk Aversion)*

Assume any monotonic increase in an independent, zero-mean background risk. Let $u'(W) > 0$ and $u''(W) < 0$. Suppose that $u'(W) \to \infty$ for $W \to \underline{W}$ and that $u'(W) \to 0$ and $r(W) \to 0$, for $W \to \infty$, where $W \varepsilon [\underline{W}, \infty)$, then

$$\frac{\partial^2 w}{\partial s \partial \phi} \geq 0, \ \forall \phi$$

and for any probability distribution of $\phi \Leftrightarrow$ utility is standard risk averse.

We first establish three lemmas which are required in the proof. We have

Lemma 1 *Suppose that $u'(W) \to \infty$ for $W \to \underline{W}$, then $r(W) \to \infty$ and $p(W) \to \infty$ for $W \to \underline{W}$.*

Proof: $u'(W) \to \infty$, for $W \to \underline{W}$, implies $\partial \ln u'/\partial W \to -\infty$, and hence $r(w) \to \infty$. Also, since for $W \to \underline{W}$, $r' < 0, p > r$, and hence $p(W) \to \infty$. □

The second lemma establishes the equivalence of declining risk aversion and declining derived risk aversion. We have:

Lemma 2 $\hat{r}'(w) \leq 0$ *for any background risk* $\Leftrightarrow r'(W) \leq 0$

Proof: Kihlstrom et. al. (1981) and Nachman (1982) have shown that declining risk aversion implies declining derived risk aversion. Conversely, declining derived risk aversion for small background risks requires declining risk aversion of $u(W)$.□

The third lemma establishes a condition for declining prudence, in the case of monotonic changes in background risk:

Lemma 3 *For monotonic increases in background risk, the marginal rate of substitution of background risk for wealth increases in ϕ, if and only if prudence declines in wealth:*

$$\frac{d}{d\phi}\left[-\frac{\partial \nu'(w)/\partial s}{\partial \nu'(w)/\partial w}\right] \geq 0 \Leftrightarrow p'(W) \leq 0$$

Proof: See Appendix 3

Proof of Proposition (3): Totally differentiating the left hand side of equation (4) with respect to s yields

$$\frac{d\nu'(w)}{ds} = \frac{\partial \nu'(w)}{\partial s} + \frac{\partial \nu'(w)}{\partial w}\frac{\partial w}{\partial s}. \tag{10}$$

It also follows from equation (4) that

$$d\nu'(w)/ds = \nu'(w)\partial \ln \lambda/\partial s.$$

Hence, the effect of the background risk on the demand for claims is given by

$$\frac{\partial w}{\partial s} = -\frac{\partial \ln \lambda}{\partial s}\frac{1}{\hat{r}(w)} - \frac{\partial \nu'(w)/\partial s}{\partial \nu'(w)/\partial w} \tag{11}$$

Sufficiency of Standard Risk Aversion First, we show $\partial ln\lambda/\partial > 0$. In order to satisfy the budget constraint, $\frac{\partial w}{\partial s}$ has to be positive in some states and negative in others. Given positive prudence, $\partial\nu'(w)/\partial s > 0$ so that the second term in (11) is positive. It follows that the first term must be negative. Since the risk aversion $\hat{r}(w)$ is positive, it follows that $\frac{\partial \ln \lambda}{\partial s}$ is positive. Second, standard risk aversion implies that $r'(w) \leq 0$ and hence by lemma 2, $\hat{r}'(w) \leq 0$. Hence, since $\frac{\partial w}{\partial \phi} < 0$, the first term in equation (11) increases in ϕ. From lemma 3, $p'(W) \leq 0$ implies also that the second term increases in ϕ.

Necessity of Standard Risk Aversion We establish necessity of standard risk aversion by considering special cases, where the probability distribution of ϕ has almost all its probability mass at a particular value $\phi = \phi_0$. Also, we consider the special case where, initially, there is no background risk, and increases in background risk are small. First, we consider the term $-\frac{\partial\lambda/\partial s}{\lambda}$. We have from equation (4)

$$E[\nu'(w)] = E[u'(w+y)] = E(\lambda\phi) = \lambda$$

and

$$\frac{\partial \lambda}{\partial s} = \frac{d}{ds}E[u'(w+y)]$$

where

$$E[u'(w+y)] = EE_y[u'(w+y)] = E[u'(w-\psi)],$$

where ψ is the precautionary premium as defined by Kimball(1990). Hence,

$$\frac{\partial \lambda}{\partial s} = E\left\{u''(w-\psi)\left[\frac{\partial w}{\partial s} - \frac{\partial \psi}{\partial s} - \frac{\partial \psi}{\partial w}\frac{\partial w}{\partial s}\right]\right\}$$

Assume that we start from a position of no background risk. In this case, $s=0$, $\psi=0$, and $\partial\psi/\partial w = 0$, hence

$$\frac{\partial \lambda}{\partial s} = E\left\{u''(w)\left[\frac{\partial w}{\partial s} - \frac{\partial \psi}{\partial s}\right]\right\} = E\left\{u''(w)\left[\frac{\partial w}{\partial s} - \frac{1}{2}p(w)\sigma^2\right]\right\}$$

where σ^2 is the variance of the small increase in background risk. In order to evaluate the derivative $\partial\lambda/\partial s$, take the case where the probability that $\phi = \phi_0 approaches 1$. In this case,

$$\frac{\partial\lambda}{\partial s} \rightarrow u''(w_0)\left[\frac{\partial w_0}{\partial s} - \frac{1}{2}p(w_0)\sigma^2\right]$$

where $w_0 = w(\phi_0)$, and hence

$$\frac{\frac{\partial\lambda}{\partial s}}{\lambda} \rightarrow \frac{u''(w_0)}{u'(w_0)}\left[\frac{\partial w_0}{\partial s} - \frac{1}{2}p(w_0)\sigma^2\right].$$

Substituting in (11), we have

$$\frac{\partial w}{\partial s} = r(w_0)\left[-\frac{\partial w_0}{\partial s} + \frac{1}{2}p(w_0)\sigma^2\right]\frac{-1}{\hat{r}(w)} - \frac{\partial\nu'(w)/\partial s}{\partial\nu'(w)/\partial w}$$

Starting with no background risk, the term

$$-\frac{\partial\nu'(\cdot)/\partial s}{\partial\nu'(\cdot)/\partial w} = \frac{1}{2}p(w)\sigma^2,$$

since $\partial\psi/\partial w = 0$ Hence we can write

$$\frac{\partial w}{\partial s} = r(w_0)\left[-\frac{\partial w_0}{\partial s} + \frac{1}{2}p(w_0)\sigma^2\right]\frac{-1}{\hat{r}(w)} + \frac{1}{2}p(w)\sigma^2$$

and

$$\frac{\partial}{\partial\phi}\left[\frac{\partial w}{\partial s}\right] = \left\{r(w_0)\left[-\frac{\partial w_0}{\partial s} + \frac{1}{2}p(w_0)\sigma^2\right]\frac{\hat{r}'(w)}{\hat{r}(w)^2} + \frac{1}{2}p'(w)\sigma^2\right\}\frac{\partial w}{\partial\phi}$$

Since $\partial w/\partial\phi < 0$, the condition for a less steep slope becomes

$$r(w_0)\left[-\frac{\partial w_0}{\partial s} + \frac{1}{2}p(w_0)\sigma^2\right]\frac{\hat{r}'(w)}{\hat{r}(w)^2} + \frac{1}{2}p'(w)\sigma^2 \leq 0. \tag{12}$$

We can now establish the necessity of declining absolute risk aversion: $r' \leq 0$ and the necessity of declining absolute prudence: $p' \leq 0$. First, suppose that $\hat{r}'(w) > 0$ for some ϕ_1, then by contradiction of the requirement in (12) we can choose $\phi_0 > \phi_1$ so that $w_0 \rightarrow \underline{W}$. Since by lemma (1), $\hat{r}(w) \rightarrow \infty$ and $p(w) \rightarrow \infty$ for $w \rightarrow \underline{W}$, inequality (12) can only be true if the first term is non-positive. Consider the bracket in the first term. In order to satisfy the budget constraint, $\frac{\partial w_0}{\partial s}$ has to be close to zero, and hence the bracket is positive. Therefore $\hat{r}'(w) > 0$ implies that the first term in (12) $\rightarrow \infty$ so that the inequality in (12) is violated. Hence $\hat{r}' \leq 0$ is necessary and by lemma (2) $r' \leq 0$. This also establishes the necessity of positive prudence, $p > 0$.

Now suppose that $p'[w(\phi_2)] > 0$ for some ϕ_2. Then, we can choose $\phi_0 < \phi_2$, so that by assumption, $r[w(\phi_0)] = r(w_0) \rightarrow 0$. $r(W) \rightarrow 0$ for $W \rightarrow \infty$ implies, together with $r'(W) =\leq 0$ that $r'(W) = r(W)[r(W) - p(W)] \rightarrow 0$ and, hence, $-r(W)p(W) \rightarrow 0$. Hence the first term in (12) approaches zero and therefore the inequality in (12) would be contradicted. Hence $p'(w) \leq 0$ is also necessary. In summary, standard risk aversion is a necessary condition for the slope of the demand function for contingent claims to decline. $\square$

Proposition (3) allows us to analyze the effect of a marginal increase in a zero-mean, independent background risk, given that this increase has a negligible impact on the prices of state-contingent claims. Proposition (3) says that an increase in s will reduce the steepness of the slope of this agent's demand function. As can be seen from Proposition (3), the agent reacts to a monotonic increase in background risk by purchasing more claims in states for which the price ϕ is high, financing the purchase by selling some claims in the states with low prices. Proposition (3) can also be interpreted by comparing, within an equilibrium, the demand of agents, who differ only in the size of their respective background risks. Proposition (3) suggests that agents with higher background risk will adjust their demand functions by buying state-contingent claims on high-price states and selling claims on low-price states. Agents with high background risk will buy "insurance" (i.e. claims on high-price states) from those with low background risk.

4 Conclusions

The main conclusions of the effects of an increase in background risk, on risk aversion and on the demand for contingent claims, are summarised in the three propositions of the paper. Proposition 1 provides a necessary and sufficient condition for simple increases in background risk to increase the derived risk aversion of agents. The condition on utility is weaker than Kimball's standard risk aversion but stronger than Gollier and Pratt's risk vulnerability. By considering only the set of simple increases in background risk we find a larger set of utility functions which satisfy the criterion of increased derived risk aversion than those of Eeckhoudt, Gollier and Schlesinger. We then proceed to examine the condition for 'generalised risk aversion', whereby agents react to increased background risk by reducing the slope of the demand curve for state contingent claims. We find in proposition 2 that increased derived risk aversion is necessary, but not sufficient, for generalised risk aversion. The stronger requirement for generalised risk aversion is shown for the case of monotonic increases in background risk in proposition 3. Standard risk aversion, i.e. positive, declining absolute risk aversion and absolute prudence, is a necessary and sufficient condition for generalised risk aversion.

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Appendix 1

Proof of Proposition (1)

From the definition of $\hat{r}(w)$,

$$\hat{r}(w) = \frac{E_y[-u''(W)]}{E_y[u'(W)]} \tag{13}$$

Differentiating with respect to s, we have the following condition:

$$\frac{\partial \hat{r}(w)}{\partial s} > [=][<]0 \iff f(w,s) > [=][<]0 \tag{14}$$

for *any* distribution of y, where $f(w,s)$ is defined as

$$f(w,s) \equiv E_y\left[y'\left\{-u'''(W) - u''(W)\hat{r}(w)\right\}\right] \tag{15}$$

with y' being the marginal change in y such that $y' \leq [=][\geq]0$ when $y < [=][>]y_0$ and $E[y'|w] = 0$ (simple increase in background risk).

Necessity

We now show that

$$\begin{aligned} f(w,s) &> [=][<] \quad 0 \Longrightarrow \\ u'''(W_2) - u'''(W_1) &< [=][>] \quad -r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \end{aligned}$$

Consider a background risk with three possible outcomes, y_0, y_1, and y_2, such that $y_0 = 0$, $y_1 < 0$, and $y_2 > 0$. Define

$$\begin{aligned} W_i &= w + y_i, \quad i = 1,2 \\ W_0 &= w. \end{aligned}$$

and let q_i denote the probability of the outcome, y_i. For the special case of such a risk, equation (15) can be written as

$$f(w,s) = q_1|y_1'|\left\{-u'''(W_2) + u'''(W_1) - [u''(W_2) - u''(W_1)]\hat{r}(w)\right\} \tag{16}$$

since

$$E[y'] = \sum_{i=0}^{2} q_i y_i' = 0$$

so that

$$q_1|y_1'| = q_2 y_2'$$

Now $\hat{r}(w)$ can be rewritten from (13) as

$$\begin{aligned}
\hat{r}(w) &= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}\frac{-u''(W)}{u'(W)}\right\} \\
&= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}r(W)\right\}
\end{aligned} \tag{17}$$

Hence, $\hat{r}(w)$ is the expected value of the coefficient of absolute risk aversion, using the risk-neutral probabilities given by the respective probabilities multiplied by the ratio of the marginal utility to the expected marginal utility. Thus, $\hat{r}(w)$ is a convex combination of the coefficients of absolute risk aversion at the different values of y. For the three-point distribution being considered, $\hat{r}(w)$ is a convex combination of $r(W_0)$, $r(W_1)$, and $r(W_2)$. Hence, as $q_0 \to 1$, $\hat{r}(w) \to r(W_0)$. Since W_0 can take any value in the range $[W_1, W_2]$, $f(w,s)$ must have the required sign for *every* value of $r(W_0)$, where $W_1 \leq W_0 \leq W_2$. Thus, since $q_1|y_1'| > 0$, this is true only if the condition in (16) holds. This is the same condition as stated in Proposition 1.

Sufficiency

To establish sufficiency we use a method similar to that used by Pratt and Zeckhauser (1987) and by Gollier and Pratt (1996).

a) We first show

$$\begin{aligned}
u'''(W_1) - u'''(W_2) &> r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \\
\Longrightarrow f(w,s) &> 0, \quad \forall (w,s)
\end{aligned} \tag{18}$$

We need to show that $f(w,s) > 0$, for all non-degenerate probability distributions. Hence, we need to prove that the minimum value of $f(w,s)$ over *all* possible probability distributions $\{q_i\}$, with $E(y') = 0$, must be positive. In a manner similar to Gollier and Pratt (1996), this can be formulated as a mathematical programming problem, where $f(w,s)$ is minimized, subject to the constraints that all q_i are non-negative and sum to one, and $E(y') = 0$. Equivalently, this can be reformulated as a parametric linear program where the non-linearity is eliminated by writing $\bar{r}$ as a parameter

$$\min_{\{q_i\}} f(w,s) = \sum_i q_i \left[y' \left\{ -u'''(W_i) - u''(W_i)\bar{r} \right\} \right] \tag{19}$$

s.t.

$$\sum_i q_i y_i' = 0 \tag{20}$$

$$\sum_i q_i = 1 \tag{21}$$

the definitional constraint for the parameter $\bar{r}$

$$\bar{r} \sum_i q_i u'(W_i) = -\sum_i q_i u''(W_i) \tag{22}$$

and the non-negativity constraints

$$q_i \geq 0, \quad \forall i \tag{23}$$

A sufficient condition for $\partial \hat{r}/\partial s > 0$ is that $f(w,s)$ as defined by (19) is positive for *any* probability distribution $\{q_i\}$ subject to $E(y') = 0$ and the definition of $\bar{r}$ given in (22).

Since we are looking for a sufficient condition for $f(w,s) > 0$, we can relax the non-negativity constraint for q_0 in the above linear program. In case even this (infeasible) resulting minimum is positive, then we know that the solution of the above linear program is always positive. We drop the non-negativity constraint on q_0, the probability of the y_0 state in the following manner. We define q_0^+ and q_0^- such that

$$q_0 = q_0^+ - q_0^- \tag{24}$$

where both q_0^+ and q_0^- are non-negative. These new variables replace q_0 in the program.

The modified linear program has three variables in the basis since there are three constraints in the program. In the optimal solution, one basis variable is either q_0^+ or q_0^-. Hence, the optimal solution of the modified linear program is (q_0, q_1, q_2) and the objective function is

$$f^*(w,s) = q_1 y_1' \left[-u'''(W_1) - u''(W_1)\bar{r} \right] + q_2 y_2' \left[-u'''(W_2) - u''(W_2)\bar{r} \right] \tag{25}$$

Since $q_1 y_1' + q_2 y_2' = 0$, it follows that (25) can be rewritten as

$$f^*(w,s) = q_1 y_1' \left[(-u'''(W_1) - u''(W_1)\bar{r}) - (-u'''(W_2) - u''(W_2)\bar{r})\right] \tag{26}$$

Hence

$$u'''(W_1) - u'''(W_2) - \left[u''(W_2) - u''(W_1)\right]\bar{r} > 0 \tag{27}$$

is a sufficient condition for $f^* > 0$, given $\bar{r}$.

As shown in equation (17), $\bar{r}$ is a convex combination of $r(W_0)$, $r(W_1)$ and $r(W_2)$, hence $\bar{r} \in \{r(W)|W \in [W_1, W_2]\}$. Hence, a sufficient condition for (27) is that

$$u'''(W_1) - u'''(W_2) - r(W)\left[u''(W_2) - u''(W_1)\right] > 0 \tag{28}$$

for all $\{W_1 \leq W \leq W_2\}$ as given by the condition of Proposition 1.

b) By an analogous argument, it can be shown that

$$\begin{aligned} u'''(W_1) - u'''(W_2) &< r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \\ \Longrightarrow f(w,s) &< 0 \quad \forall (w,s) \end{aligned} \tag{29}$$

c) We now show directly that

$$\begin{aligned} u'''(W_1) - u'''(W_2) &= r(W)\left[u''(W_2) - u''(W_1)\right], \forall W_1 \leq W \leq W_2 \\ \Longrightarrow f(w,s) &= 0 \quad \forall (w,s) \end{aligned} \tag{30}$$

A sufficient condition for $f(w,s) = 0, \forall (w,s)$ is that $\min_{\{q_i\}} f(w,s) = \max_{\{q_i\}} f(w,s) = 0$, subject to (20)-(22) and the nonnegativity condition for every q_i except q_0. The minimum and maximum involve three basis variables, one of which is either q_0^+ or q_0^-. Therefore, $f^*(w,s)$ is always determined by (26). Hence, the minimal and maximal value of $f^*(w,s)$ are zero if the bracketed term in (26) is zero. This is the case if

$$u'''(W_1) - u'''(W_2) = r(W)\left[u''(W_2) - u''(W_1)\right], \quad \forall W_1 \leq W \leq W_2. \tag{31}$$

□

Appendix 2

Proof of Lemma (2)

We have to show that

$$-\frac{\partial \nu'(w)/\partial s}{\partial \nu'(w)/\partial w}$$

increases in ϕ, if and only if absolute prudence is declining. This term increases in ϕ if the negative of the term increases in w since $\partial w/\partial \phi < 0$. In terms of the underlying utility function, this is the same as showing that the term

$$Z(w) = \frac{E_y[u''(W)y']}{E_y[u''(W)]}$$

is increasing in w, where $y' = \partial y/\partial s$.
Now consider a marginal increase in w. Then

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } E_y[u''(W)]E_y[u'''(W)y'] - E_y[u'''(W)]E_y[u''(W)y'],$$

or,

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } - E_y[\{u'''(W) - u''(W)\frac{E_y[u'''(W)]}{E_y[u''(W)]}\}y'],$$

and then it follows that

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } - E_y[\{u'''(W) - u''(W)\frac{E_y[u'''(W)]}{E_y[u''(W)]}\}(y' - \hat{y}')],$$

where $\hat{y}' = y'(\hat{y})$ and $\hat{y}$ is defined by

$$\hat{y} = y|\frac{E_y[u'''(W)]}{E_y[-u''(W)]} - p(W) = 0.$$

Hence

$$\text{sign } \frac{\partial Z(w)}{\partial w} = \text{sign } E_y[-u''(W)\{\frac{E_y[u'''(W)]}{E_y[-u''(W)]} - p(W)\}(y' - \hat{y}')]. \qquad (32)$$

Monotonic increases in background risk imply that

$$y' - \hat{y}' \leq [=][\geq]0, \text{for} y < [=][>]\hat{y}.$$

It follows that absolute prudence, $p(W)$, must be declining if sign $\frac{\partial Z(w)}{\partial w}$ is to be non-negative for any distribution of y, in particular for any binomial distribution. Sufficiency of declining prudence for sign $\frac{\partial Z(w)}{\partial w}$ to be non-negative follows from

$$\{\frac{E_y[u'''(W)]}{E_y[-u''(W)]} - p(W)\}(y' - \hat{y}') \geq 0, \forall y.$$

Hence, declining absolute prudence is necessary and sufficient. □